THE ANALYTICAL DEVELOPMENT OF
CURVES AND STREAMLINE SHAPES

by

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PREFACE

The procedures and methods presented herein are intended primarily to serve as a guide and text to those whose work consists of developing smooth, continuous lines with a uniformly changing rate of curvature. These requirements are met in many engineering fields of endeavor, including the development of aircraft and automobile shapes, ship lines, projectile and missile shapes and trajectories, highway and railroad curves, and many other similar branches of engineering.

The increased speeds required of airplanes, ships, missiles, railroads, etc., - to mention only a few cases - necessitate extreme care being paid to the design of their external shapes so that proven drag penalties, both in operating cost and performance, may be minimized. Hence it is no longer suffic--ient, or advisable, to resort to the old method of drawing these lines "by eye," but instead these lines must be developed from analytical expressions which will insure that uniform rates of change in curvature are obtained, and that even the minutest reversals in curvature are eliminated. Both of the above condit--ions must be satisfied in order to keep the drag of any given shape as small as possible.

In the further case of railroad track and highway layout, the smooth transition from a straight track, or highway, into a curve and then back again into a straight course, requires that these transitions be started and ended gradually, and that the rate of change of curvature vary smoothly along the entire length. Here again analytical expressions defining these curves are needed to insure that these requirements are satisfied.

The use of such analytical expressions for the shapes of many bodies

also permits the easy calculation of the volumes, surface areas, etc., of these bodies, and allows proportional changes in shape, volume, or area, to be quickly made. This reduces considerably the time required making successive layouts, each time checking to determine if the desired conditions have been obtained.

The development of equations for such purposes as outlined above, and for many other similar cases which will suggest themselves to the reader, is given in detail herein. The equations have been kept as simple as possible, with only elementary algebra and calculus employed, so that it is not necessary for the reader to be well versed in either of these subjects in order to make use of the equations, or develop those of his own. The text, however, will serve as a handy guide and reference to elementary calculus, as it represents a current practical application of the calculus.

These methods of analytically developing curved lines and streamlined shapes have been employed for many years in the design of aircraft manufactured by the Republic Aviation Corporation, whose airplanes have long been recognized for their exceptionally high performance. No small amount of this performance gain is due to the extreme care taken in the analytical development of the shapes involved, so that smooth airflow may be maintained, and drag losses kept to a minimum.

In this regard, the author wishes to express his deep appreciation of the efforts of Mr. A. Kartveli, Chief Engineer of Republic Aviation Corporation, whose insistence upon perfection in design led to the need for the development of the methods and equations given on the following pages. Grateful acknowledg--ment is also extended to Dr. W. J. O'Donnell and Messrs. Hoshen Lu and J. Cravero, of Republic Aviation Corporation, for their aid and recommendations in the preparation of the manuscript.

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FOREWORD

The generation of faired, curved surfaces is a problem frequently encountered in such engineering applications as ship building, airplane construct--ion, automobile body design, etc. This generation is usually performed graphically, and is based on the geometrical properties of known curves, usually of the second degree. In many cases, surfaces are even generated without any reference to known curves simply by "feel and eye".

In the course of the last few years, however, the rigid requirements of aeronautical design made it necessary to devise a much more accurate analytical method for the generation of curved surfaces. This method consists in performing the work of generation purely analytically by representing various curves evolved by algebraic equations. By a combination of such equations, desired shapes can be obtained with a great degree of accuracy. The use of equations, in addition, enables the computation of areas, volumes, centers of gravity, etc., to be considerably simplified.

Mr. Harry Haase has been engaged in this type of work for several years at the Republic Aviation Corporation, and this book represents the results of his experience. His method of expressing the ordinates of a curve as a function of the abscissae, by means of a polynomial with fractional exponents, proves to be very successful in generating wings and streamlined bodies, and in fairing the intersect--ion between two bodies. In fact, the efficiency of the method is so great that it has now obsoleted and replaced other methods previously used by the Engineering Department of the Republic Aviation Corporation.

This book will bring a valuable contribution to the Engineering liter--ature on this subject, and should prove a very useful guide to designers engaged in this type of work.

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CHAPTER I

THE CIRCLE AND THE ELLIPSE

1.00 - Use of the Circle in the Development of Contour Lines:- In the development of contour lines for bodies moving through a fluid, such as air or water, two types of lines are of prime interest to the designer. These are the horizontal plane contour lines, or lines parallel to the direction of fluid flow which are usually called streamlines, and the lines perpendicular to the direction of fluid flow, or cross-sectional lines. Contour lines in other planes are generally not necessary when the above-mentioned contour lines are thoroughly and accurately developed.

The circle is of particular importance in the development of streamlines , or cross-sectional contour lines. While it is seldom, if ever, used directly as a streamline, it is often used in the development of certain airplane wing airfoil sections, and in laying out the leading edges of streamlined shapes. As a cross section, the circle is most efficient for a body moving through a fluid, and is extensively used for airplane fuselages, rockets, missiles, pressurized vessels, etc.

A shape with circular cross sections along its entire length is known as a "body of revolution", which is formed by rotating any line, curved or straight, 360 degrees about its reference axis. Hence, rotating a straight line completely about a reference axis parallel to it generates a cylinder, the simplest type of body of revolution, while rotating, for example, the upper contour line of an airfoil about its reference axis will produce a streamlined body of revolution.

If a volume of fluid is forced past a body of revolution parallel to its reference axis, or if the body is pushed or pulled through a stationary fluid, the fluid displaced by this body will move, at a given cross section

or station, at the same speed, and the pressure will be uniform around the periphery for each station. Therefore, no cross-flow can result at any station. The fluid disturbances thus created will be small, and the drag produced will then be a minimum.

In the case of any other cross section, such as an elliptical one, for example, the fluid flowing along the major axis will travel farther than that moving along the minor axis, since the distance is longer due to the increased curvature. The fluid will then increase in speed along the minor axis to keep pace with the free fluid, but will increase in speed more along the major axis for the same reason. A difference in pressure will then result around the periphery at a given station, and a cross-flow created to cause fluid disturbances and consequently more drag. The ellipse, however, has its place in the development of various contour lines, as will be shown later.

1.10 - Equations of the Circle:-

1.11 - Origin at Center:- The Pythagorean theorem states that, in a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two other sides. By use of this theorem, it is possible to locate any point on the periphery of a circle of a fixed radius if either the abscissa or the ordinate is known, or, if the radius is not known, it can be determined when the abscissa and ordinate of a point are given.

In the circle shown in Fig. 1, let (p) be any point on the periphery

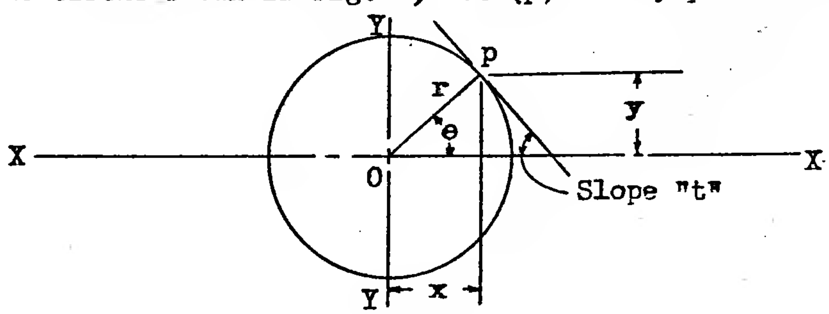


FIGURE 1

at a distance (x) from the vertical reference line, or Y axis, and distance (y) from the horizontal reference line, or X axis, and let (r) be the radius.

Then, from the above theorem:

$$y^2 + x^2 = r^2 \quad \text{.............................(1)}$$

or:

$$y = \sqrt{r^2 - x^2} \quad \text{.............................(2)}$$

The slope at any point on the periphery may be found by means of the first derivative of Eq. (1):

$$2y(dy/dx) + 2x = 0$$

$$(dy/dx) = -(x/y) \quad \text{...........................(3)}$$

The second derivative of Eq. (1), which is often used in line development work for the purpose of determining points of inflection, or change in curvature, is found by differentiating the first derivative as given in Eq. (3):

$$\begin{aligned}(d^2y/dx^2) &= (x/y^2)(dy/dx) - 1/y \\ &= (x/y^2)(-x/y) - 1/y \\ &= -(x^2/y^3 + 1/y) \\ &= -(1/y^3)(x^2 + y^2) \quad \text{.........................(4)}\end{aligned}$$

Substituting from Eq. (1):

$$(d^2y/dx^2) = -(r^2/y^3) \quad \text{...........................(4a)}$$

Very often it is easier to use the trigonometric equations of a circle in layout work, such as when laying out circles of large radius. In Fig. 1, for example, it is evident that:

$$(y/r) = \sin\theta$$

$$y = r(\sin\theta) \quad \text{.................................(5)}$$

and:

$$(x/r) = \cos\theta$$

$$x = r(\cos\theta) \quad \text{.................................(6)}$$

From Eqs. (5) and (6) the ordinates of a circle of any given radius (r) can be easily plotted by assuming values of 0 and computing, with the aid of trigonometric tables, the corresponding values of (x) and (y).

1.12 - Transfer of Origin:- In the previous equations the origin of the circle was assumed to be at the center. It is more convenient, though, in some instances to have the origin on the X axis and on the periphery, as at point (B) in Fig. 2. This is particularly true when another curve is added to an arc of a circle, as in the case of a leading edge of an airfoil section. All abscissae can then be measured from one point, the transferred origin of this circle.

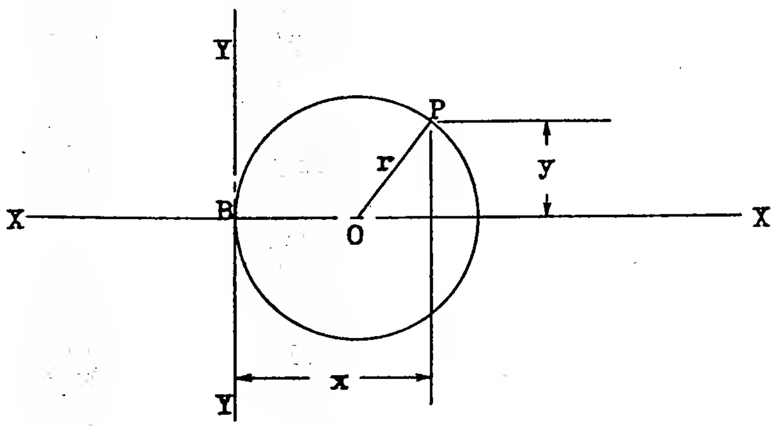


FIGURE 2

Since the distance from the center of the circle, (O), to point (B) is equal to the radius (r), the Pythagorean theorem of Eq. (1) can be written:

$$y^2 + (x - r)^2 = r^2$$

or:

$$y^2 + x^2 - 2rx + r^2 = r^2$$

$$y^2 = 2rx - x^2 \quad \text{.................................(7)}$$

$$y = \sqrt{2rx - x^2} \quad \text{.................................(7a)}$$

The first derivative, or slope at any point on the periphery, is:

$$(dy/dx) = \frac{(r - x)}{\sqrt{2rx - x^2}} \quad \text{.........................(8)}$$

Substituting $y = \sqrt{2rx - x^2}$ from Eq. (7a) in Eq. (8), the slope is:

$$(dy/dx) = (r - x)/y \qquad (9)$$

The second derivative is obtained as follows by differentiating Eq. (9):

$$(d^2y/dx^2) = \left[(-1)(r - x)y^{-2}\right](dy/dx) - y^{-1}$$

Rearranging, and substituting the value of (dy/dx) from Eq. (9):

$$\begin{aligned}(d^2y/dx^2) &= -((r - x)/y^2)((r - x)/y) - (1/y)\\ &= -(r - x)^2/y^3 - (y^2/y^3)\\ &= (-(r - x)^2 - y^2)/y^3 \qquad (10)\end{aligned}$$

Eqs. (7a), (9), and (10) only apply when the origin is placed on the X axis at the periphery of the circle, as shown in Fig. 2. In this form, the development of faired lines can be most easily accomplished, and the equations are simpler to use. If the origin of the circle should be displaced both from the X axis and the periphery, as in Fig. 3, the general equation of the circle becomes:

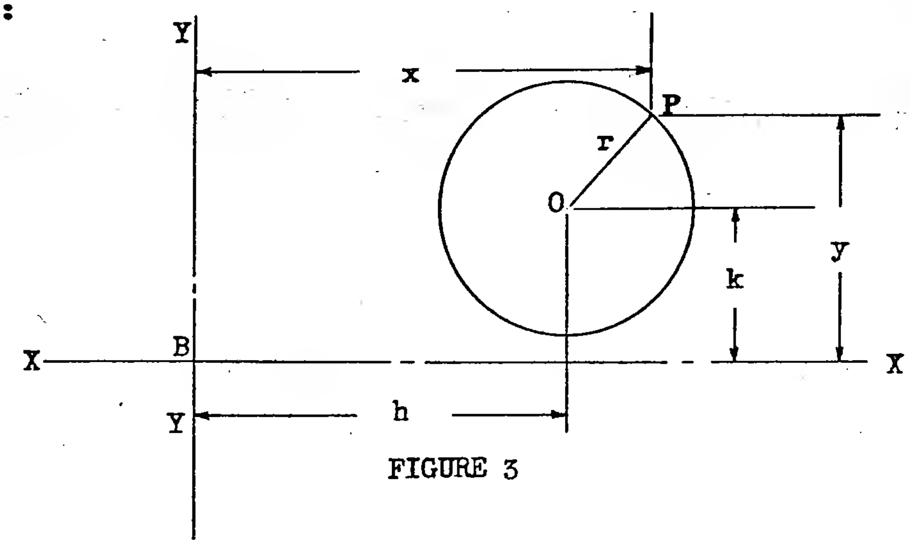


FIGURE 3

$$(x - h)^2 + (y - k)^2 = r^2 \qquad (11)$$

The derivatives of Eq. (11) become considerably more complicated to use than those of Eq. (7a), and since it is seldom, if ever, necessary in line development work to transfer the origin in such a manner as shown in Fig. 3, Eq. (11) need not be further discussed.

Transfer of the origin to a point not on the periphery, but on the X axis, is, however, useful when it is necessary to add to the left hand side of the circle. This is shown in Fig. 4 below:

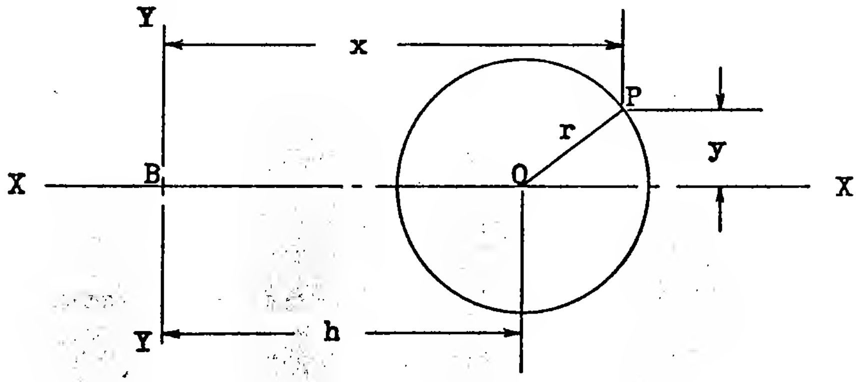


FIGURE 4

In this case, it is readily seen that the equation of the circle is expressed as:

$$y^2 + (x - h)^2 = r^2 \quad \text{.............................(12)}$$

or:

$$y^2 + x^2 - 2hx + h^2 = r^2$$

and solving the above for (y):

$$y = \sqrt{r^2 - x^2 - h^2 + 2hx} \quad \text{.........................(13)}$$

1.13 – <u>Polar Coordinates of the Circle</u>:– The equations of a circle can also be expressed in polar coordinate form, instead of the rectangular coordinate fo~n previously discussed. This is shown in Fig. 5, where the origin, or pole, is assumed to be on the periphery of the circle and also on the X axis.

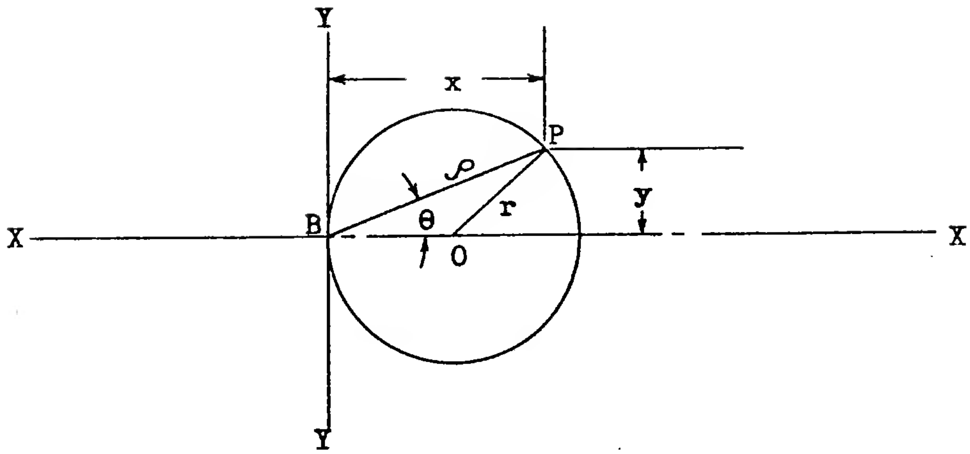


FIGURE 5

Again using the theorem that the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides:

$$\rho^2 = x^2 + y^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

and, from trigonometry:

$$x = (\rho)\cos(\theta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

The basic expression for transfer of the origin is given by Eq. (7), which may also be written:

$$x^2 + y^2 = 2rx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Hence, if $(\rho)^2$ is substituted in Eq. (16) for $(x^2 + y^2)$, and $\cos\theta$ for (x), the polar equation of a circle with the origin on the X axis and on the periphery becomes:

$$\rho^2 = 2r(\cos\theta)$$

?

$$\text{or:} \quad \rho = 2r(\cos\theta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

For polar coordinates of a circle with the origin any distance (h) from the center, as shown in Fig. 6, but still on the X axis, Eqs. (14) and (15) still apply.

Substituting the values of $(x^2 + y^2)$ and (x) from Eqs. (14) and (15), respectively, in the general equation of a circle with the origin transferred

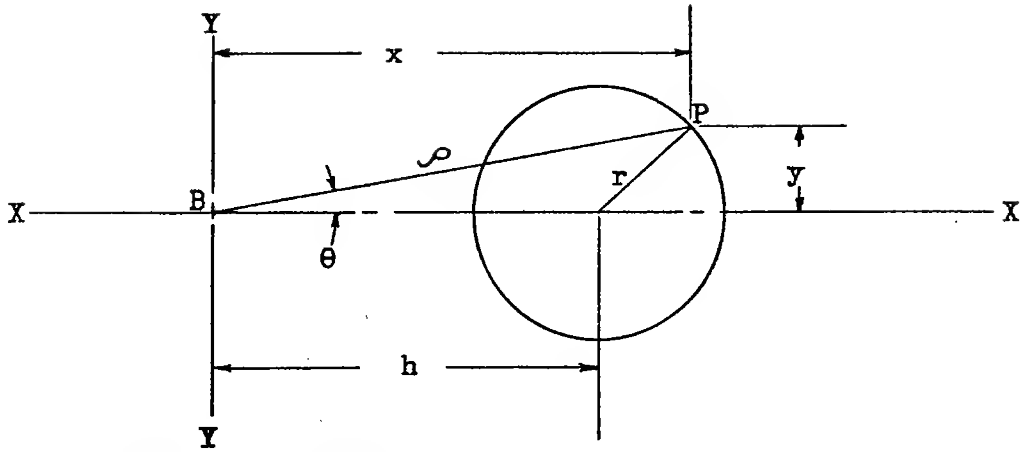


FIGURE 6

any distance (h) along the X axis, (Eq. (12a)), we obtain the final equation:

$$\rho^2 = r^2 - h^2 + 2h\rho \cos \theta \quad \text{..........................(18)}$$

1.20 - Adaptation of the Ellipse to Development of Contour Lines:- The ellipse is one of the most widely used geometric figures in the development of contour lines, being extensively used for cross-sections and segments of streamlined shapes.

As a cross-section of an airplane fuselage, for example, it has not only been used for convenience in designing the encloseure of a cockpit, leaving ample and suitable space in the fuselage for fuel tanks, air ducts, controls, equipment, and the like, but it has a definite advantage over a circular cross-section in the case of a mid-wing airplane, where the wing panels join the fuselage. It was stated before that the fluid flowing along the major axis of an ellipse has to travel a longer distance than the fluid moving along the minor axis. It will, therefore, have to move faster along the top of the fuselage, (it is assumed that the major axis of the ellipse is vertical), to keep up with that flowing along the sides.

However, the wing too has thickness, and the fluid travelling along the minor axes of the ellipses of the fuselage will be forced to go over or under the wing surfaces with increased velocity. The fluid speeds will thus be

equalized, to some extent, in the region up to the maximum thickness of the airfoil section. From there on the air will expand, and the ratio of major to minor axes of the ellipses should be gradually reduced until they finally end up in a circle at the rear of the fuselage. If the horizontal tail surfaces are attached directly to the fuselage, an elliptical cross-section should be maintained, since the same velocity changes will occur.

In aircraft work, another item of interest may be noted with regard to the location of the wings with respect to the fuselage. The upper surface of conventional airfoil sections usually has more curvature than the lower surface, and therefore a higher fluid velocity occurs over the upper surface. This velocity differential, when expressed by means of Bernoulli's equation in terms of its corresponding pressure differential, is necessary for the creation of lift. To more nearly equalize the fluid velocities over the wing and fuselage, therefore, it is advisable to locate the wing slightly below the centerline of the fuselage. This helps to preserve a smooth flow at the junction, and has been found by experience to materially reduce the so-called "interference" drag. Obviously, the amount of displacement of the wing from the fuselage centerline should be determined from a consideration of whether the fuselage cross-section is circular, elliptical, or otherwise, and also on a consideration of the airfoil section being used.

1.30 – Equations and Construction of the Ellipse:-

1.31 – Standard Equation of the Ellipse:- The ellipse, which in reality is a flattened circle with uniformly varying curvature, can be constructed by the use of two circles with a common center at the origin. The radius of the larger circle becomes the major axis of the ellipse, while the radius of the smaller circle is the minor axis.

Vectors passing through the origin may then be drawn at random. For

each vector a line can be drawn parallel to the X axis passing through the intersection of this vector with the periphery of the small circle. Similarly, for this same vector, a line can be drawn parallel to the Y axis passing through the intersection of this vector with the periphery of the larger circle. The intersection of these two lines then determines one point on the periphery of the ellipse, such as point (p) on Fig. 7. Continuing this process, as many points as desired can be found, and joined together to form the ellipse.

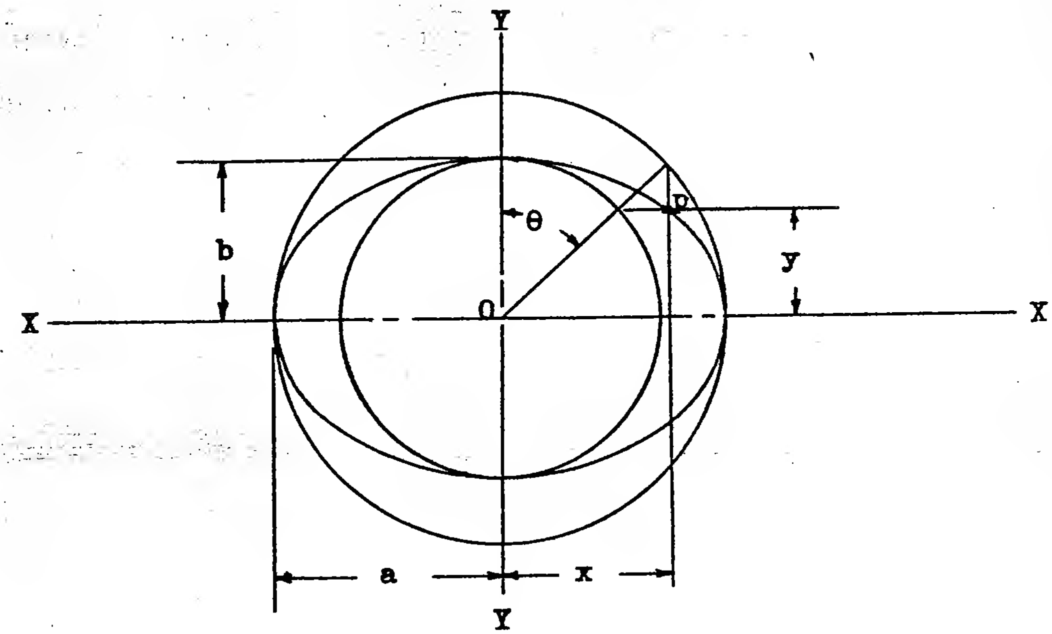


FIGURE 7

It is evident from Fig. 7 that:

$$\sin \theta = (x/a) \quad \text{..................................(19)}$$

and:

$$\cos \theta = (y/b) \quad \text{..................................(20)}$$

Using the relationship ($\sin^2 + \cos^2$) = 1, from trigonometry, Eqs. (19) and (20) may be combined to form the standard equation of an ellipse:

$$(x/a)^2 + (y/b)^2 = 1$$

or:

$$x^2/a^2 + y^2/b^2 = 1 \quad \text{..........................(21)}$$

Eq. (21) can be solved for (y) as follows:

$$y^2 = b^2((1 - (x^2/a^2))$$
$$= (b^2/a^2)(a^2 - x^2)$$

$$\text{or:} \quad y = (b/a)\sqrt{a^2 - x^2} \quad \text{............................(22)}$$

The slope of the curve at any point on the periphery is determined, as in the case of the circle, by finding the first derivative of the equation for the curve. Hence, using Eq. (21) and differentiating:

$$2x/a^2 + 2y(dy)/b^2(dx) = 0$$

from whence:

$$(dy)/(dx) = -(b^2x)/(a^2y) \quad \text{........................(23)}$$

The second derivative, which is used for determining the rate of change of slope, or the locations of any points of inflection, is found as follows:

$$(dy)/(dx) = -(b^2x)/(a^2y)$$
$$= -(b^2x)(a^2y)^{-1}$$

Then:

$$(d^2y)/(dx^2) = (-b^2x)(-1)(a^2y)(a^2)((dy)/(dx)) + (a^2y)^{-1}(-b^2)$$
$$=((b^2xa^2)/(a^2y)^2)(dy/dx) - (b^2)/(a^2y)$$
$$= \frac{b^2xa^2(dy/dx) - b^2(a^2y)}{(a^2y)^2}$$

Substituting the value of (dy/dx) from Eq. (23):

$$(d^2y)/(dx^2) = \frac{b^2xa^2(-b^2x/a^2y) - b^2(a^2y)}{a^4y^2}$$
$$= -b^2(b^2x^2 + a^2y^2)/(a^4y^3)$$

If Eq. (21) be multiplied by (a^2b^2), there results $(b^2x^2) + (a^2y^2)$ (a^2b^2). Therefore, substituting this equation in the above:

$$(d^2y)/(dx^2) = -((b^2(a^2b^2))/(a^4y^3)$$
$$= -(b^4)/(a^2y^3) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24)$$

Eq. (24) then gives the value of the second derivative of the standard equation of an ellipse.

1.32 - Plotting an Ellipse by Means of Trigonometric Functions:- A convenient method of plotting an ellipse is by means of trigonometric tables. By assuming any value of (x), and letting (a) be the major axis, the sine of the angle can be computed by employing Eq. (19). The cosine can then be interpolated from trigonometric tables, and (y) found by multiplying the cosine by (b), the minor axis. This establishes a point on the periphery. Repeating this process for other values of (x), a number of points can be plotted, and connected by means of a spline or french curves. The angles are not needed in degrees, and do not have to be noted.

An example of the method of computing the ordinates of an ellipse with a major axis of (a) = 10 and minor axis of (b) = 6 is shown in Table I:

TABLE I

x	$(x/a) = \sin\theta$	$\cos\theta$	$y = (b)\cos\theta$
2	0.200	0.979	5.874
4	0.400	0.916	5.496
5	0.500	0.866	5.196

Plotting the results, a curve as shown in Fig. 8 will be obtained:

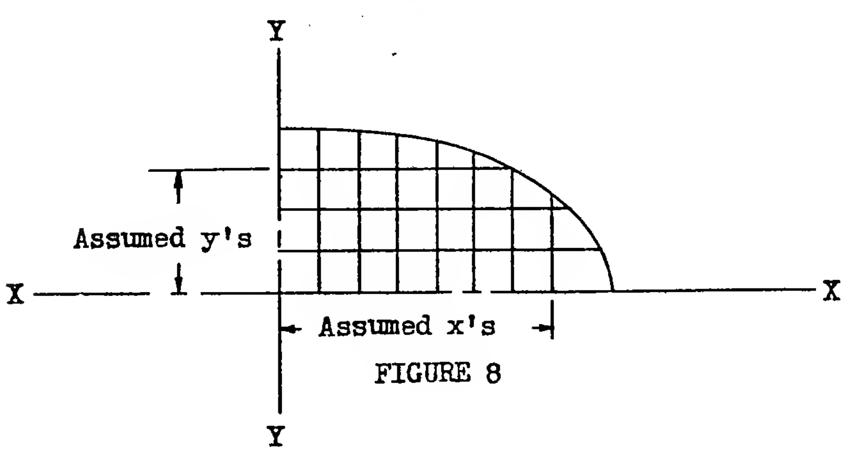


FIGURE 8

It will be found that it becomes quite difficult to plot an accurate ellipse when the angle θ reaches its maximum. Therefore, the process of computing the ordinates can be reversed by assuming values of (y) and then calculating the corresponding values of (x). It is advisable, for a good check, to compute the (y's) up to about 40 degrees and then start with values of (x) from about 30 degrees so that the points overlap. The Table will be the same as Table I except that the first column will be (y), the second $(y/b) = \cos\theta$, the third $\sin\theta$, and the fourth $(x) = (a)\sin\theta$. It is obvious that only one quadrant need be computed, as the others can be copied.

1.33 - Transfer of Origin:- Just as in the case of a circle, it is often more convenient to transfer the origin to a point on the X axis and on the periphery of the ellipse.

Transferring the origin from (O) to (B), as shown in Fig. 9, the distance between these two points is equal to (a), the major axis of the

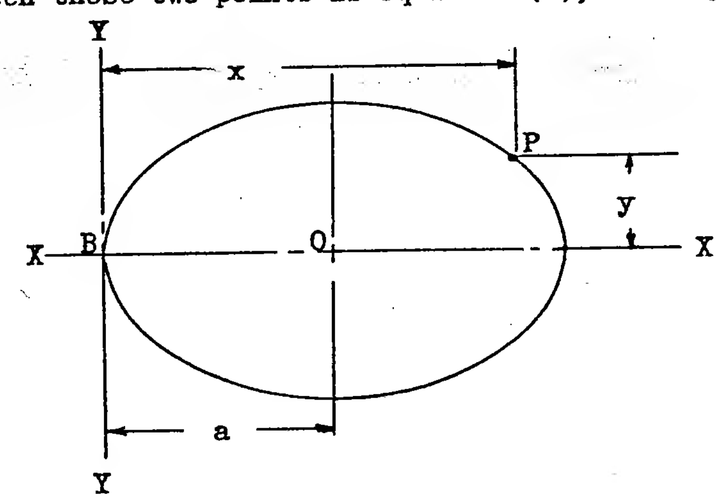


FIGURE 9

ellipse. Employing the standard equation of the ellipse, as given by Eq. (21), and substituting (x - a) for (x), we get:

$$(x - a)^2/a^2 + y^2/b^2 = 1$$

$$(x^2 - 2ax + a^2)/a^2 + y^2/b^2 = 1$$

Multiplying both sides of the equation by a^2b^2:

$$b^2(x^2 - 2ax + a^2) + a^2y^2 = a^2b^2$$

or:

$$y^2 = (2ab^2x - b^2x^2)/a^2$$
$$= (b^2/a^2)(2ax - x^2)$$

and:

$$y = (b/a)\sqrt{2ax - x^2} \qquad \text{..........................(25)}$$

It will be noted that Eq. (25) becomes the same as Eq. (7a), the equation of a circle with its origin transferred as in this case, when $b = a = r$, the radius of a circle.

The slope at any point on the periphery, (dy/dx), is found as follows:

From Eq. (25):

$$y = (b/a)(2ax - x^2)^{\frac{1}{2}}$$

Then:

$$(dy/dx) = (b/a)(\tfrac{1}{2})(2ax - x^2)^{-\frac{1}{2}}(2a - 2x)$$
$$= ((b(a - x))/ a\sqrt{2ax - x^2}$$

Substituting the value of $\sqrt{2ax - x^2}$ from Eq. (25):

$$(dy/dx) = b(a - x)/(a)(ay/b)$$
$$= b^2(a - x)/a^2y \qquad \text{..............................(26)}$$

The second derivative in this case is:

$$(d^2y)/(dx^2) = (b^2/a^2)\left[(a - x)(-y^{-2}(dy)/(dx)) + (y^{-1} - 1)\right]$$
$$= (b^2/a^2)\left[-((a - x)(dy/dx))/y^2 - 1/y\right]$$

Substituting the value of (dy/dx) from Eq. (26):

$$(d^2y)/dx^2 = -(b^2/a^2)\left[((a - x)(b^2/a^2)\ ((a - x)/y))/y^2 + 1/y\right]$$
$$= -(b^2/a^2)\left[\frac{(a^2b^2 - 2ab^2x + b^2x^2) - a^2y^2}{a^2y^3}\right]$$
$$= \frac{-a^2b^4 + 2ab^4x - b^4x^2 - a^2b^2y^2}{a^4y^3} \qquad \text{.........................(27)}$$

By letting $a = b = r$, the radius of a circle, in Eq. (27), the expression for the second derivative of the equation of a circle with its origin transferred to the periphery on the X axis, Eq. (10), is obtained, as would be expected.

1.34 – <u>Polar Coordinates of an Ellipse</u>:– The polar coordinates of an ellipse are sometimes used in layout work, although perhaps not as often as the rectangular coordinates due to slightly more required computation. Once the necessary calculations have been made, however, the actual layout is somewhat simpler as it is only needed to mark off various distances on a given set of radii.

In Fig. 10, below, we can see that:

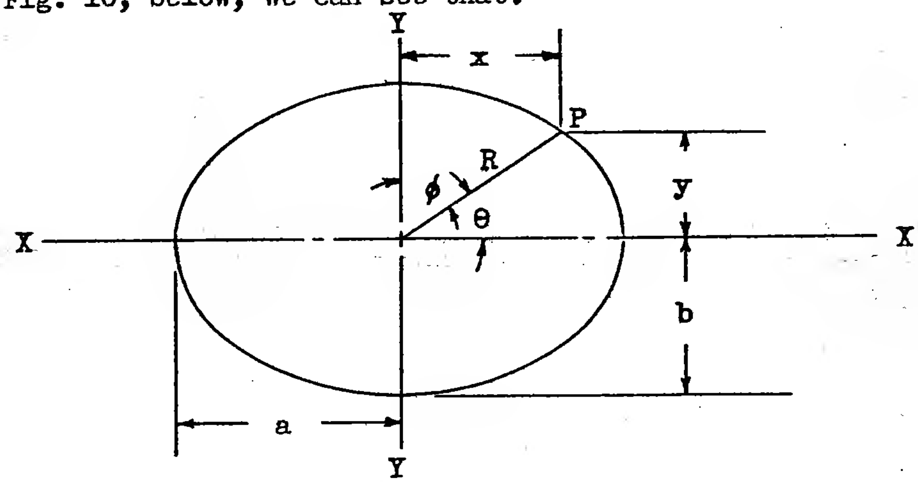


FIGURE 10

$$x = R(\sin \phi)$$

and:

$$y = R(\cos \phi)$$

where (R) is the radius to any given point (P) on the periphery of the ellipse.

If these values be substituted in Eq. (21), the standard equation of an ellipse in rectangular coordinates, then:

$$(R(\sin \phi))^2/a^2 - (R(\cos \phi))^2/b^2 = 1$$

or:

$$R^2(\sin^2\phi/a^2 - \cos^2\phi/b^2) = 1$$

This can be reduced to:

$$R = (ab)/\sqrt{b^2\sin^2\phi - a^2\cos^2\phi} \quad \text{.......................(28)}$$

Eq. (28) then gives an expression for the radius to any point on the periphery of the ellipse in terms of the major and minor axes, (a) and (b), and the angle (ϕ) as defined in Fig. 10.

For the angle θ, it can be shown in a similar manner as above that:

$$R = (ab)/\sqrt{b^2\cos\theta - a^2\sin\theta} \quad \text{.....................(29)}$$

The development of the equations for the first and second derivatives of Eqs. (28) and (29) shall not be attempted here, as they become too complex to be practical for layout work. When the need arises for such expressions, it is simpler to convert to the rectangular coordinate system, or to the use of power curves which will be outlined in the next chapter.

1.40 - Other Basic Geometric Shapes:- Other basic geometric shapes, such as the parabola, hyperbola, etc., could also be used in developing contour lines for various shapes of bodies. However, by using the power curves of the next chapter, the circle and the ellipse can be so easily modified to fit any desired contour that there is no need, at this time, to consider any other basic geometric figures. Later on a discussion is given of some applications of the parabola, which may be used sometimes to advantage in the development of equations.

CHAPTER II

THE POWER CURVES

2.10 - Introduction:- Power curves are so called because the powers, or exponents, of the dependent variables of the equations are the main factors that govern the shape of the curves. The equations may be single or multi-term, and each consists essentially of four parts; the dependent variable, the independent variable, the constant, and the exponent. When either of the latter two is unity it can be, and usually is, omitted in the equation. Any symbols may be used for the component parts, but to follow the general practice of mathematical study, it shall be assumed that (y) is the dependent variable, (x) the independent variable, (k) the constant, and (n) the exponent, or power. Thus a single term power equation will be written:

$$y = k(x)^n \quad \text{..............................(30)}$$

The dependent variable (y) is a function of (x), and can be computed for any (x) desired. It represents the ordinate, while (x) is the abscissa.

The constant (k) governs the height, or thickness, of the curve, and is usually small when the exponent is large, and large when the power is small. A change in magnitude of the constant will increase or decrease all (y's) in the same proportion.

The exponent, (n), which governs the curvature, may be divided into three classes. First, the exponent can be equal to unity. A term with this power will form a straight line, the constant of which determines the slope. Second, the exponent can be larger than unity. A term with this power will form a curve that follows the X axis at the origin for an infinitesimal distance, turns up swiftly, and continues with ever-decreasing curvature. Third, the exponent may be smaller than unity. A term with this power will form a curve

that follows the Y axis at the origin for an infinitesimal distance, turns swiftly to the right, and then also continues with ever-decreasing curvature. This exponent is usually required for the first term of all equations of streamlined shapes developed by this method.

If Eq. (30) is evaluated, it will be found that, except at the origin, the slope of the curve will never be zero or unity, no matter how far it is extended. The curve, therefore, will never be parallel to either the X or Y axes. To obtain this, however, one or more terms may be added or subtracted. Thus, if a negative term with a small constant and an exponent larger than unity, (if that of the first term is less than unity), is added, the curve will become parallel to the X axis at some point and will then eventually intersect it. If a positive term is added under the same conditions, however, the curve will not become parallel to the Y axis, as will be shown in a later chapter.

An equation with one or more terms added to the first is called a multi-term equation. If subscripts corresponding to the term number are used for (k) and (n), a multi-term equation will then be:

$$y = k_1(x)^{n_1} - k_2(x)^{n_2} - k_3(x)^{n_3} \text{ ...etc.} \qquad (31)$$

The slope at any point on the curve can readily be determined by means of the first derivative, which is:

$$(dy/dx) = n_1k_1(x)^{n_1-1} - n_2k_2(x)^{n_2-1} - n_3k_3(x)^{n_3-1} \text{...etc.} \qquad (32)$$

It can be easily seen that Eq. (32) is in the same form as Eq. (31), since the values (n_1k_1), (n_2k_2), etc., are also constants. Similarly, the equation for the second derivative is expressed in the same form. These equation forms, while they may seem involved at first glance, are simple and have an unparalleled flexibility ideal for layout work.

It has been found that it is easier to deal with power curve equations with fractional exponents when common logarithms are employed. Therefore, in the discussions to follow on the use of power curves, the constants are left in log form and noted with the letter (N), signifying that the number following is not the actual value of the constant but the logarithm of the constant. For example, a single term equation with a constant of 3 and an exponent of 0.5 would be written:

$$y = N(0.47712)(x)^{0.5}$$

For evaluation of (y) in the above equation, the log of (x) is multiplied by the exponent 0.5 and added to 0.47712. The antilog of the sum, as found from logarithmic tables, then equals the value of (y).

In the case of an exponent of unity, there is no need to resort to the use of logarithms, and the actual values of the constants can be more readily used.

2.20 - The Circle and the Ellipse in Power Curve Form:- The use of power curves, and a better understanding of their form, can best be shown by first working with the equations of the circle and the ellipse as derived in Chapter I.

Consider the equation of the ellipse with the origin on the X axis and on the periphery, as given by Eq. (25). This can be written:

$$y^2 = (b^2/a^2)(2ax - x^2)$$

or, multiplying out the right hand side:

$$y^2 = (2b^2x)/(a) - (b^2x^2)/(a^2)$$

The right hand side of this equation can now be put in the form of the right hand side of Eq. (31), or in power curve form, by letting:

$$k_1 = (2b^2)/(a) \qquad \text{(32a)}$$

$$k_2 = (b^2)/(a^2) \qquad \text{(32b)}$$

Then the equation may be written:

$$y^2 = k_1(x) - k_2(x)^2$$

or, solving for (y):

$$y = (k_1x - k_2x^2)^{\frac{1}{2}} \quad \text{..........................} (33)$$

Thus Eq. (33) represents the power curve form of Eq. (25). In this form, as will be seen later, the equation becomes more flexible for use in modifying the basic elliptical shape into contour lines of different curvatures.

The slope at any point on the curve of this equation can be found, as before, by means of the first derivative. Hence:

$$(dy)/(dx) = \tfrac{1}{2}(k_1x - k_2x^2)^{-\frac{1}{2}}(k_1 - 2k_2x)$$

$$= \frac{(k_1 - 2k_2x)}{2(k_1x - k_2x^2)^{\frac{1}{2}}}$$

Substituting (y) for $(k_1x - k_2x^2)^{\frac{1}{2}}$, from Eq. (33):

$$(dy)/(dx) = (k_1 - 2k_2x)/(2y) \quad \text{......................} (34)$$

By substituting in Eq. (34) the values of (k_1) and (k_2), as given above, it will be found that it is the same equation as Eq. (26), but that it is in a somewhat simpler form.

The second derivative is found as follows:

$$(dy)/(dx) = \tfrac{1}{2}(k_1x - k_2x^2)^{-\frac{1}{2}}(k_1 - 2k_2x)$$

Then, differentiating the above:

$$d^2y/dx^2 = \tfrac{1}{2}\left[(k_1x - k_2x^2)^{-\frac{1}{2}}(-2k_2)\right] + \tfrac{1}{2}\left\{\left[k_1 - 2k_2x\right]\left[-\tfrac{1}{4}(k_1x - k_2x^2)^{-3/2}(k_1 - 2k_2x)\right]\right\}$$

$$= \frac{-2k_2}{2(k_1x - k_2x^2)^{\frac{1}{2}}} + \frac{(k_1 - 2k_2x)^2}{4(k_1x - k_2x^2)^{3/2}}$$

$$= \frac{-\ 4k_2(k_1x - k_2x^2) + (k_1 - 2k_2x)^2}{4(k_1x - k_2x^2)^{3/2}}$$

$$= \frac{-4k_2(k_1x - k_2x^2) + (k_1 - 2k_2x)^2}{4(k_1 - k_2x^2)(k_1x - k_2x^2)^{\frac{1}{2}}}$$

Substituting (y) for $(k_1x - k_2x^2)^{\frac{1}{2}}$, from Eq. (33), and (y^2) for $(k_1x - k_2x^2)$, the final equation in power curve form becomes:

$$(d^2y/dx^2) = \frac{-4k_2y^2 + (k_1 - 2k_2x)^2}{4y^3} \quad \text{.....................(35)}$$

2.21 - Combining Part of an Ellipse with an Arc of a Circle:- As an example of the way in which the power curve form of the equation of an ellipse is used, consider the case where it is necessary to add part of an ellipse to the arc of a circle. This can be done mathematically for layout work, and should be done in this manner to assure a smooth continous curve, with a uniformly changing rate of curvature.

In Fig. 11, assume that the radius of the circle is (r), the distance

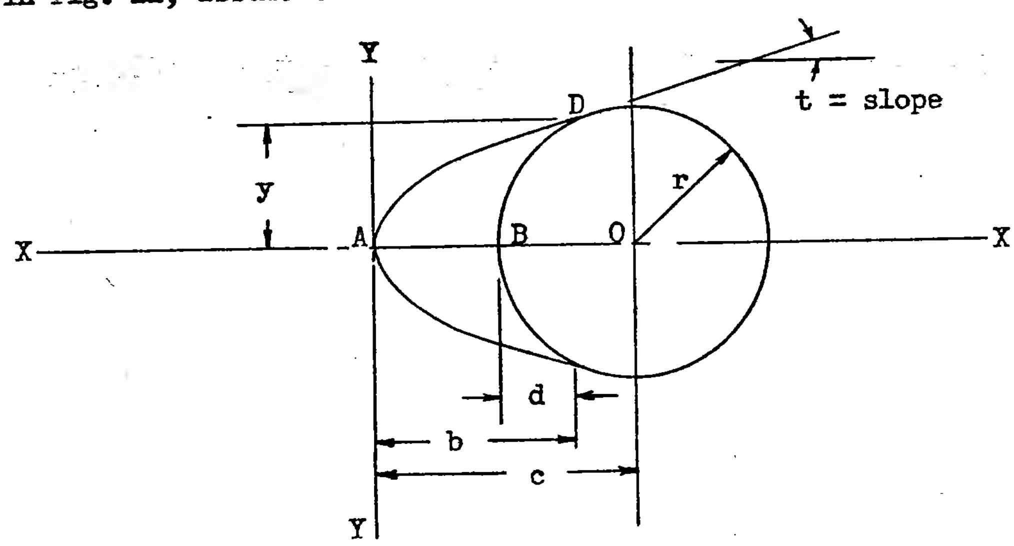


FIGURE 11

from the center line to the maximum width required is (c), the distance from the periphery of the circle to the point of tangency is (d), and the distance from the maximum width desired to the point of tangency is (b).

Eq. (7a) gives the equation of a circle with the origin at (B), as in Fig. 11.

$$y = (2rx - x^2)^{\frac{1}{2}} \quad \text{.............................(7a)}$$

The slope at any point is given by Eq. (9):

$$(dy/dx) = (r - x)/y \quad \text{.............................(9)}$$

Using Eq. (7a), the value of (y) at the point of tangency, (D), can be found, and Eq. (9) permits the slope to be found at the same point. Having found these values, the equation for an ellipse passing through point (D) with the given values of (y) and (dy/dx) can now be developed. Using preferably the power curve form of the equations for an ellipse with the origin at (A), as in Fig. 11, the right hand side of Eq. (33) is equated to the right hand side of Eq. (7a), and also the right hand sides of Eqs. (34) and (9) can be equated.

Hence, for the ordinate at the point of tangency:

$$(2rx - x^2)^{\frac{1}{2}} = (k_1x - k_2x^2)^{\frac{1}{2}}$$

or, using the symbols as in Fig. 11, and cancelling the root signs:

$$(2rd - d^2) = (k_1b - k_2b^2) \quad \text{.........................(36)}$$

Similarly for the slope:

$$(r - x)/y = (k_1 - 2k_2x)/2y$$

or:

$$2(r - d) = (k_1 - 2k_2b) \quad \text{.........................(37)}$$

Eqs. (36) and (37) can now be solved simultaneously to determine the values of k_1 and k_2, and these values then put in the form of Eq. (33) to give the required equation of the ellipse.

An example is given below to illustrate the method outlined. Assume, in Fig. 11, that the following quantities are known or given:

$$b = 35$$
$$c = 40$$
$$d = 25$$
$$r = 30$$

Using Eq. (36):

$$2(30)(25) - (25)^2 = k_1(35) - k_2(35)^2$$

or:

$$25 = k_1 - 35(k_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(38)$$

Using Eq. (37):

$$2(30 - 25) = k_1 - 2k_2(35)$$

or:

$$10 = k_1 - 70(k_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(39)$$

Solving Eqs. (38) and (39) simultaneously for k_1 and k_2 by first multiplying Eq. (39) by 2, we get:

$$50 = 2(k_1) - 70(k_2)$$

$$\underline{10 = (k_1) - 70(k_2)}$$

$$40 = (k_1)$$

Substituting this value of k_1 in Eq. (38), (or Eq. (39) could also be used), the value of k_2 is found to be:

$$25 = 40 - 35(k_2)$$

$$k_2 = 15/35$$

$$= 0.42857$$

Hence, the equation of the required ellipse, in power curve form, is:

$$y = (40b - 0.42857b^2)^{\frac{1}{2}}$$

or, since (b) really represents the conventional (x) term:

$$y = (40x - 0.42857x^2)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(40)$$

This is then the equation of an ellipse with the origin on the X axis and on the periphery, (point (A) in Fig. 11), which is also tangent with the periphery of the given circle at point (D). The slope at any point on this ellipse is then obviously the first derivative of Eq. (40).

2.22 - <u>Combining Two Ellipses</u>:- Many times it may be necessary to enlarge the cross section of elliptical bodies, with the least amount of rework

of existing parts and structure. Such may be the case with an airplane fuselage, for example, when more space is required without changing the major fuselage longerons, or affecting too much of the internal layout.

The simplest way of doing this is to add a semi-ellipse, with a larger major axis, to the ellipse on hand. In Fig. 12, below, let it be assumed that it is desired to add a portion of an ellipse to the original elliptical cross-section so as to increase the height by 10 inches. (For the sake of maintaining the same equations as previously developed, the "height" in this case is measured horizontally along the X axis. Obviously, for such layout work, it is immaterial in what direction the "height" is considered).

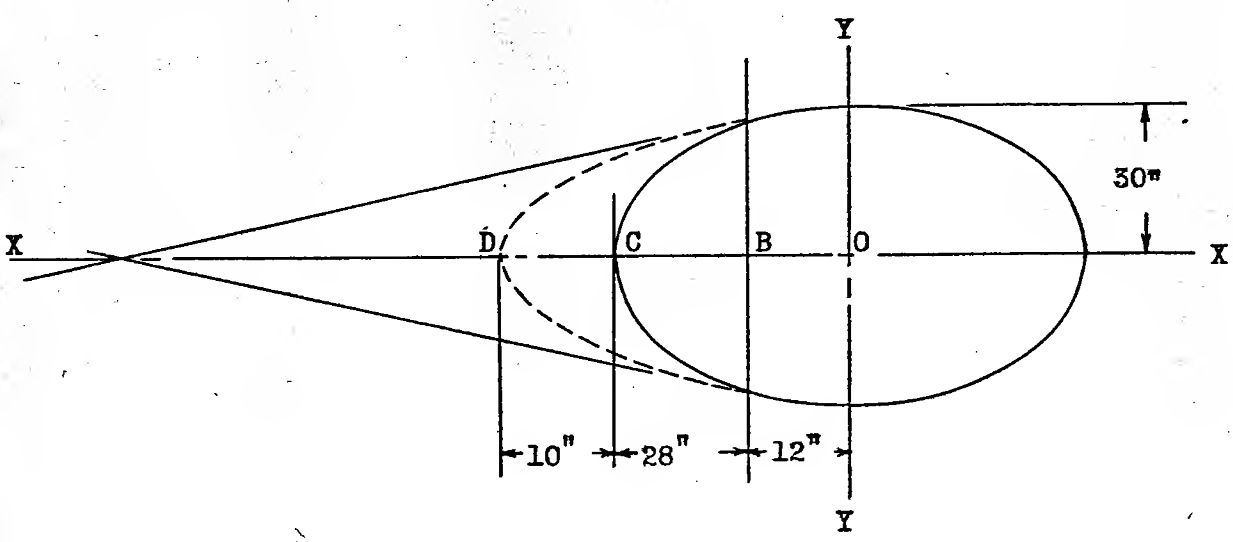


FIGURE 12

The known, or given, dimensions of the original and desired cross-sections are indicated on Fig. 12. Since the (y) values and slopes of both ellipses must be the same at Station (B) in order for them to be tangent, it is only necessary to equate the respective "curve" and "slope" equations for the two ellipses. Therefore, letting k_1' and k_2' be the constants for the original ellipse, and k_1'' and k_2'' for the segment to be added, we have, from Eq. (33):

$$(k_1'x - k_2'x^2)^{\frac{1}{2}} = (k_1''x - k_2''x^2)^{\frac{1}{2}} \quad \text{.......................(41)}$$

and, from Eq. (34):

$$(k_1' - 2k_2'x)/2y = (k_1'' - 2k_2''x)/2y \quad \text{.......................(42)}$$

It must be remembered that the values of (x) are not the same when used for the original and new ellipses, since the original ellipse has its origin at (C), and the new segment to be added has its origin at (D).

The constants of the original ellipse can be found from Eqs. (32a) and (32b):

$$k_1' = (2b^2)/a = 2(30)^2/40 = 45$$

$$k_2' = (b^2)/(a^2) = (30)^2/(40)^2 = 0.5625$$

Substituting these in Eq. (41), and remembering that the value of (x) to the point of tangency for the original ellipse is 28 inches:

$$\left[45(28) - 0.5625(28)^2\right]^{\frac{1}{2}} = (38k_1'' - 1444k_2'')^{\frac{1}{2}}$$

or:

$$k_1'' - 38k_2'' = 21.5526 \quad \text{.........................(43)}$$

Substituting the values of k_1' and k_2' in Eq. (42):

$$45 - 2(0.5625)(28) = k_1'' - 2(38)k_2''$$

or:

$$k_1'' - 76k_2'' = 13.5 \quad \text{.........................(44)}$$

The value of k_2'' can now be found by solving Eqs. (43) and (44) simultaneously, giving:

$$k_2'' = 0.2119$$

k_1'' can now be found by substituting this value of k_2'' in either Eq. (43) or (44):

$$k_1''= 29.604$$

The power curve equation of the new segment of the ellipse, with its origin at (D) and tangent to the original ellipse at Station (B), is then:

$$y = (29.604x - 0.2119x^2)^{\frac{1}{2}}$$

2.30 - <u>Development of a Single Term Power Curve Equation</u>:- The previous sections have shown how the power curve equations can be used in the case where it is required to pass the curve of an ellipse through a point, the ordinate and slope of which are given. It shall now be shown how the equation for any type of a curve can be developed which will pass through one or more given points, and, at first, the single term equation will be considered.

The standard single term power curve equation has already been shown in Eq. (30) to be of the form:

$$y = k(x)^n \quad \text{.................................} (30)$$

The parabola is an excellent example of the above form of equation. The standard parabolic equation is:

$$y^2 = 2px \quad \text{.................................} (45)$$

where (p) equals twice the distance from the focus to the vertex, or origin. The above equation may be written:

$$y = (2p)^{\frac{1}{2}}(x)^{\frac{1}{2}}$$

and, if (k) be substituted for $(2p)^{\frac{1}{2}}$;

$$y = k(x)^{0.5} \quad \text{.................................} (46)$$

This is a single term power curve equation of the same form as Eq. (30), and is often used, as is, for the first term of the multi-term equations to be developed later.

Referring to Fig. 13, let it be assumed that it is required to pass a curve through the two points (P) and (Q), whose ordinates and abscissae are as shown.

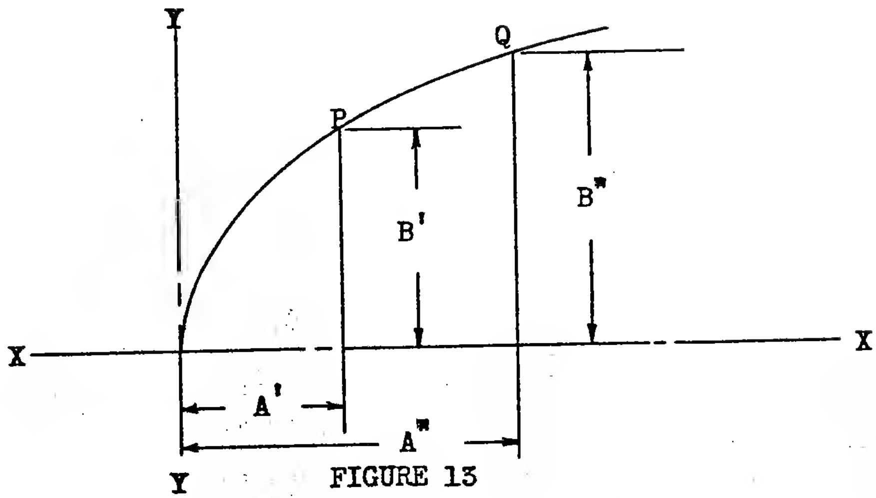


FIGURE 13

If it first be assumed that a separate equation, in the form of Eq. (30), be written for point (P) and also for point (Q), these equations become:

$$B' = k(A')^n \quad \text{.........................(47)}$$

$$B'' = k(A'')^n \quad \text{.........................(48)}$$

Then, from Eq. (47):

$$k = (B')/(A')^n \quad \text{.........................(49)}$$

and, from Eq. (48):

$$k = (B'')/(A'')^n \quad \text{.........................(50)}$$

Since it is now desired to combine Eqs. (47) and (48) into a single equation, there can be only one constant, or, in other words, the values of (k) in both equations must be one and the same. We may then write:

$$(B')/(A')^n = (B'')/(A'')^n$$

or:

$$(A''/A')^n = (B''/B')$$

Transposing, and using logarithms, the value of (n) is found to be:

$$n = \log(B''/B')/\log(A''/A') \quad \text{.........................(51)}$$

If this value of (n) is now substituted back in either Eq. (49) or

(50), the value of (k) can be found, and then the equation is developed in the familiar form:

$$y = k(x)^n$$

Often, however, it happens that a second point on the required curve is not known, or is of no importance, but instead the slope of the curve at a point is known. Then the derivative of the equation is employed for this point. For example, in Fig. 14, let (A) be the abscissa and (B) the ordinate of the point (P), and let (t) be the slope at this point.

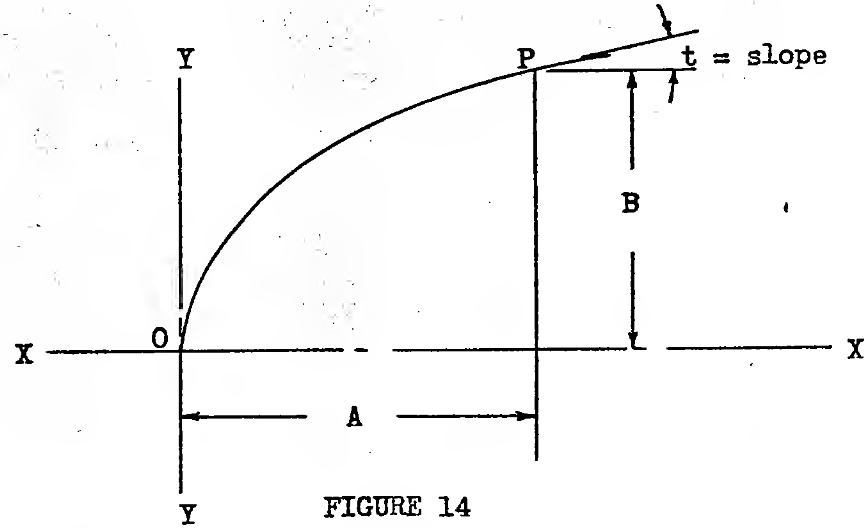


FIGURE 14

Substituting in the standard Eq. (30) these values, we get:

$$B = k(A)^n \quad \text{.................................(52)}$$

or:

$$k = B/(A)^n \quad \text{.................................(53)}$$

The first derivative of Eq. (52) is obviously:

$$(dy/dx) = (dB/dA) = t = nk(A)^{n-1}$$

The value of (k) is given by Eq. (53), and hence:

$$t = n(B/A^n)(A)^{n-1}$$

$$= nB/A$$

and then:

$$n = At/B \quad \text{.................................(54)}$$

Thus, knowing the values of (k) and (n) from Eqs. (53) and (54), respectively, the standard single term power curve equation can be set up.

Should the slope, (t), be of no importance, then the exponent (n) can be assumed and (k) found from Eq. (53) and both values used in Eq. (30) to develop the curve.

Eqs. (49) or (50), (51), (53), and (54) for the most part form the basis for the development of power curve equations, and should be memorized.

2.31 - <u>Application of a Single Term Power Curve Equation</u>: - For the purpose of demonstrating the use of the equations developed in the preceeding section, let it be assumed that it is required to form an equation for a curve passing through two given points. Therefore, in Fig. 13, assume the following to be given:

$$A' = 15$$
$$B' = 10$$
$$A'' = 32$$
$$B'' = 14$$

First substitute in Eq. (51) to find the exponent needed:

$$n = \log(14/10)/\log(32/15)$$

$$= \frac{\log(14) - \log(10)}{\log(32) - \log(15)}$$

$$= (1.14613 - 1.000)/(1.50515 - 1.17609)$$

$$= 0.44408$$

The value of (k) is found from Eq. (53); the single primed values will be used, although the double primed ones could be used just as well:

$$k = (B')/(A')^{0.44408}$$

or, using logarithms for ease of computation:

$$k = \log(B') - 0.44408(\log A')$$

$$= \log(10) - 0.44408(\log 15)$$

$$= 1.0000 - 0.44408(1.1761)$$

$$= 0.47772$$

The required equation is then:

$$y = N(0.47772)(x)^{0.44408}$$

As has been noted previously, the reader should be careful to remember that, in the above equation, the letter (N) signifies that the real value of the constant is not 0.47772, but is the number whose logarithm is 0.47772. The only reason the equation is left in this form is to facilitate computation of various values of (y) for assumed values of (x). For example, if (x) = 10, then:

$$\begin{aligned} y &= N(0.47772) + 0.44408(\log 10) \\ &= N(0.47772) + 0.44408(1) \\ &= N(0.9218) \end{aligned}$$

The number whose logarithm is 0.9218 is found from log tables to be 8.35, and hence:

$$y = 8.35 \text{ when } x = 10$$

As a second example, consider the case where it is desired to find the equation of a curve which passes through a point at a given slope, (t). Referring to Fig. 14, let:

$$\begin{aligned} A &= 20 \\ B &= 10 \\ t &= 0.2 \end{aligned}$$

From Eq. (54):

$$\begin{aligned} n &= At/B \\ &= (20)(0.2)/10 \\ &= 0.4 \end{aligned}$$

and, from Eq. (53):

$$\begin{aligned} k &= B/(A)^n \\ &= 10/(20)^{0.4} \\ &= N\left[\log(10) - 0.4\log(20)\right] \\ &= N(0.47959) \end{aligned}$$

Hence, the required equation is:

$$B = N(0.47959)(A)^{0.40}$$

or, in usual terms:

$$y = N(0.47959)(x)^{0.40}$$

The equation for the curve can easily be checked for any error as follows. Substituting A = 20, then:

$$\begin{aligned} B &= N(0.47959)(20)^{0.40} \\ &= N(0.47959 + 0.40 \log (20)) \\ &= N(0.47959 + 0.52041) \\ &= N(1.0000) \\ &= 10.000, \text{ which checks.} \end{aligned}$$

To check the slope, consider the first derivative of the equation:

$$\begin{aligned} (dy/dx) = (dB/dA) = t &= (0.4)\left[N(0.47959)\right](x)^{-0.6} \\ &= N(\log(0.4) + 0.47959 - 0.6 \log (20)) \\ &= N\left[(0.60206-1) + 0.47959 - 0.6(1.30103)\right] \\ &= N(0.30103-1) \\ &= 0.2, \text{ which checks.} \end{aligned}$$

With variable slopes, the number of curves possible through two points, or, as in this case, the origin and a point, is infinite. For each assumed exponent, (n), the constant can be solved for by means of Eq. (53). To show, and get acquainted with, the general trend of these curves, a series of equations has been developed and the ordinates computed and plotted.

In Table II, the equations are given for curves passing through the origin and the point A = 40, B = 40, for different assumed values of the exponent, (n). The values of (k) were found from Eq. (53), and then put in log form.

TABLE II

Curves Through Origin and Point (A = 40, B = 40)

Assumed Exponent	k	Equation
0.3	N(1.12144)	$y = N(1.12144)(x)^{0.3}$
0.5	N(0.80103)	$y = N(0.80103)(x)^{0.5}$
0.7	N(0.48062)	$y = N(0.48062)(x)^{0.7}$
0.9	N(0.16021)	$y = N(0.16021)(x)^{0.9}$
1.0	N(o.00000)	$y = x$
2.0	N(0.39794-2)	$y = N(0.39794-2)(x)^{2}$
4.0	N(0.19382-5)	$y = N(0.19382-5)(x)^{4}$

Table III shows the solution for values of (y), in the equations given in Table II, for assumed values of (x). These equations are then plotted on Fig. 15.

TABLE III

Curves Through Origin and Point (A = 40, B = 40)

	(y) for an exponent (n) of:						
x	0.3	0.5	0.7	0.9	1.0	2.0	4.0
1	13.227	6.325	3.024	1.446	1.000	0.025	0.000
2	16.284	8.944	3.024	2.699	2.000	0.100	0.000
5	21.436	14.142	9.330	6.156	3.000	0.625	0.010
10	26.390	20.000	15.157	11.487	10.000	2.500	0.156
20	32.490	28.285	24.623	21.436	20.000	10.000	2.500
30	36.693	34.641	32.704	30.876	30.000	22.500	12.656
40	40.000	40.000	40.000	40.000	40.000	40.000	40.000

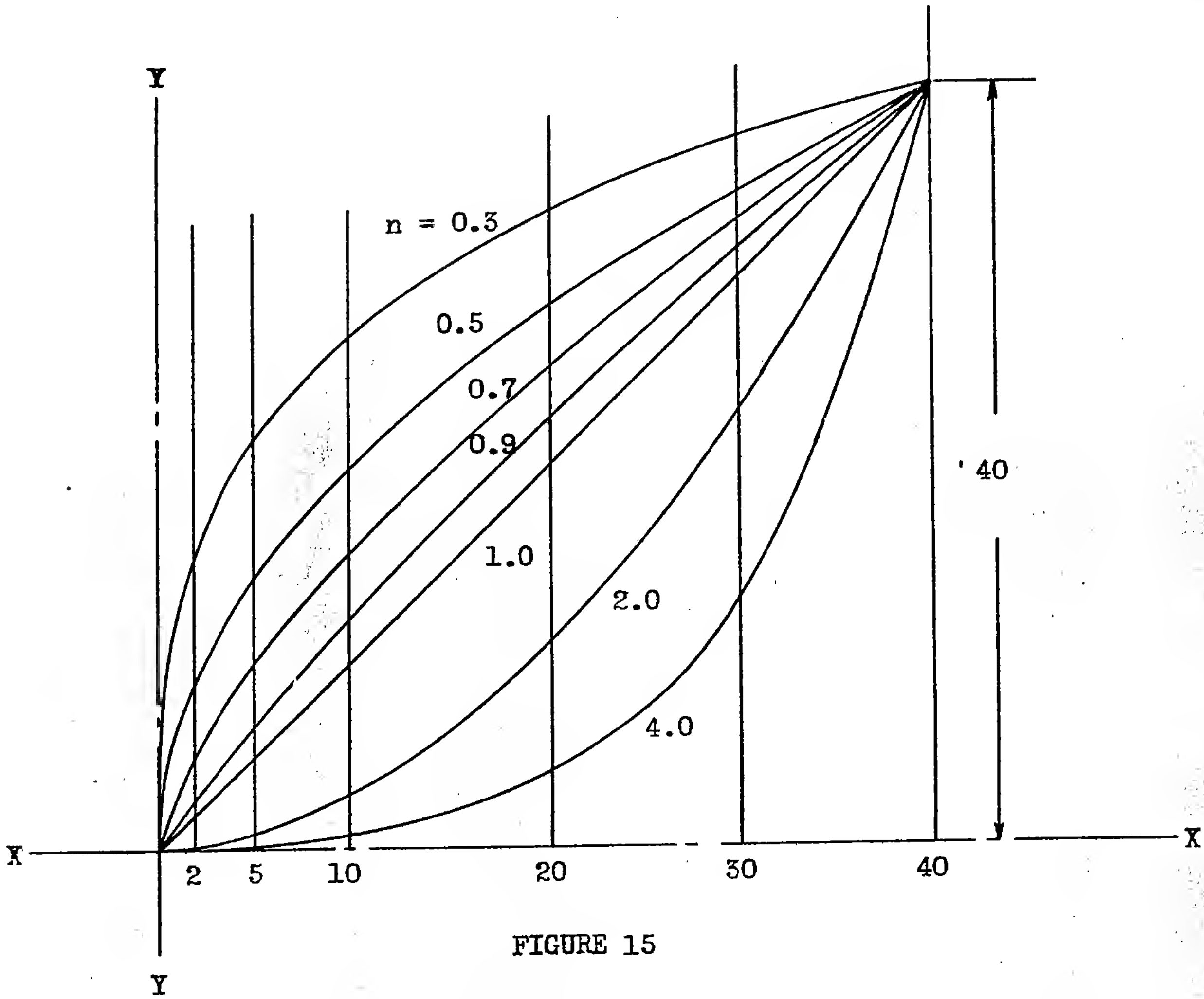


FIGURE 15

For the second example, equations are developed for curves drawn through the origin and the point A = 100, B = 40. Table IV shows the equations developed in the same manner as in the previous example.

The ordinates for the equations in Table IV are given in Table V, and the results are plotted on Fig. 16. A comparison with Fig. 15 shows clearly that, for the same exponent, the curves have the same general trend, those of Fig. 16 being flatter due to the greater (A) distance. The reader will find it time well spent to study these curves and become familiar with the way in which the shape of the curves of a single term power curve equation may be modified by using various values of the exponent.

TABLE IV

Curves Through Origin and Point (A = 100, B = 40)

Assumed Exponent	k	Equation
0.3	N(1.00206)	$y = N(1.00206)(x)^{0.3}$
0.5	N(0.60206)	$y = N(0.60206)(x)^{0.5}$
0.7	N(0.20206)	$y = N(0.20206)(x)^{0.7}$
0.9	N(0.80206-1)	$y = N(0.80206-1)(x)^{0.9}$
1.0	N(0.60206-1)	$y = N(0.60206-1)(x)$
2.0	N(0.60206-3)	$y = N(0.60206-3)(x)^2$
4.0	N(0.60206-7)	$y = N(0.60206-7)(x)^4$

TABLE V

Curves Through Origin and Point (A = 100, B = 40)

	(y) for an exponent (n) of:						
x	0.3	0.5	0.7	0.9	1.0	2.0	4.0
5	16.280	8.940	4.910	2.700	2.000	0.100	0.000
10	20.050	12.650	7.980	5.040	4.000	0.400	0.004
20	24.680	17.890	12.970	9.400	8.000	1.600	0.064
40	30.390	25.300	21.060	17.530	16.000	6.400	1.024
60	34.320	30.980	27.980	25.260	24.000	14.400	5.184
80	37.410	35.780	34.220	32.720	32.000	25.600	16.390
100	40.000	40.000	40.000	40.000	40.000	40.000	40.000

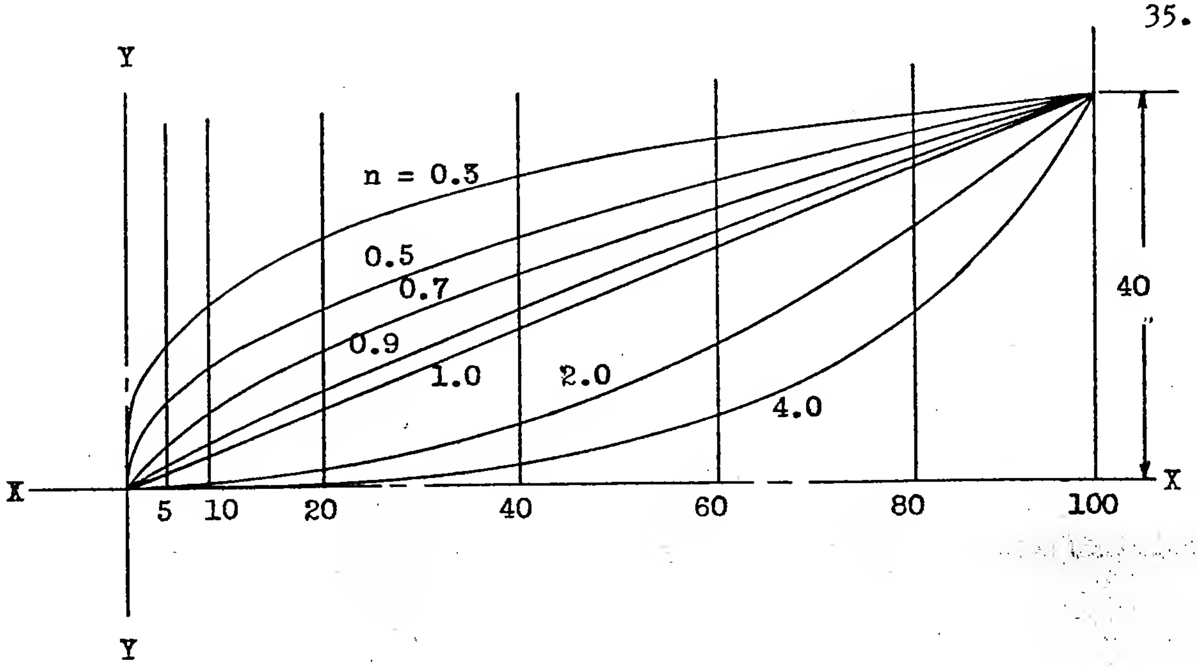


FIGURE 16

2.40 - Two Term Power Curve Equations: - It has been stated previously that a single term power curve can not be made to parallel either the X or Y axis, (except for a minute distance at the origin), and therefore can not be made to intersect either of these axes.

To obtain a curve that will parallel either the X or Y axis at some point, and later intersect the axis, one or more terms must be added to the first term. The equation is then called a multi-term equation. The constants of all these terms can be solved simultaneously, while the required exponents either have to be assumed or found by a trial and error method.

The two term equations are by far the simplest of the multi-term type, as they require little work not only in their development but also in the computation of the ordinates. However, two term equations can only be directly developed when certain information is given, and, when this information is not available, resort must be made to the trial and error method outlined in Section 2.60.

An analysis of two term equations shows that the exponents are the

only factors that govern the location of the maximum ordinate, and that the constants control the thickness ratio, where thickness ratio is defined as the ratio of the maximum ordinate to the distance from the origin to the point of intersection with the axis.

The following sections demonstrate the manner in which two term equations can be developed directly.

2.41 - Determination of the Constants of a Two Term Power Curve Equation When the Exponents are Known: - The simplest condition for which a two term power curve equation can be written is when both exponents are given, or assumed. Such conditions arise in practice through experience with the type of curves obtained with various exponents, it very often being desired to duplicate the general shape of a curve previously developed but instead passing it through two different points.

However, only two points on the desired curve can be handled in developing the two term equation, since a requirement for passing the curve through three or more points necessitates the use of three or more term equations.

The constants of an equation of two terms can be solved simultaneously in the same manner as the exponent and constant of a single term equation. It will be demonstrated here, by means of a numerical example, a method of solving for the constants of a two term equation, the exponents being assumed as 0.5 and 1.0 for the first and second terms, respectively. The curve is required to start at the origin and pass through the points (x) = 25, (y) = 12, and (x) = 100, (y) = 0. This can be assumed to be a typical example of developing the equation for an entire streamline shape which starts at the X axis and intersects the X axis again at a point 100 units away.

Eq. (31) shows the standard form of the multi-term power curve equation, and therefore, substituting the coordinates of the two points to

form the two preliminary equations, we get:

$$y = k_1(x)^{n_1} - k_2(x)^{n_2} \qquad (31)$$

$$12 = k_1(25)^{0.5} - k_2(25)^{1.0}$$

and:

$$0 = k_1(100)^{0.5} - k_2(100)^{1.0}$$

Solving these two equations simultaneously by first extracting the roots and multiplying the first equation by 2 and subtracting it from the first, (k_2) is found to be:

$$k_2 = 0.48$$

$$= N(0.68124-1)$$

The value of (k_1) is found by substituting this value of (k_2) back in either one of the above equations:

$$k_1 = 4.80$$

$$= N(0.68124)$$

The desired equation is then, from Eq. (31):

$$y = N(0.68124)(x)^{0.5} - N(0.68124-1)(x)$$

2.42 - <u>Determination of the Exponents of a Two Term Power Curve Equation</u>: - Two term power curve equations can be readily written for the condition where the value and location of the maximum ordinate , and the length along the X axis from the origin to the point of intersection, are given. This condition often occurs, as it actually consists of developing a streamline shape of a given length, thickness ratio, and position of maximum thickness.

In this case, curves are developed in this section to permit the exponent of the second term to be found directly, after the first term exponent has been assumed from a consideration of the type of curve desired. (See Figs. 15 and 16).

As mentioned in Section 2.40, the exponents of a multi-term power curve equation govern the location of the maximum ordinate. The location of the maximum ordinate along the axis is usually expressed as a percent of the distance from the origin to the point of intersection of the curve with the axis. Thus, in airplane airfoil section layout work, for example, the distance from the origin to the intersection of the curve with the X axis is called the "wing chord", and the maximum ordinate, or airfoil thickness, is located at some percentage of this chord from the origin, or leading edge.

To show how the exponents of the terms control the location of the maximum ordinate, consider the following two term power curve equation, with the constants equal to unity:

$$y = (x)^{0.5} - (x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(55)$$

From calculus, it is known that if the first derivative of an equation is set equal to zero, and solved, the value of the dependent variable, (y in this case), is a maximum or minimum. Hence, taking the derivative of Eq. (55):

$$(dy/dx) = 0.5(x)^{-0.5} - 1$$

and, letting this equal zero, the value of (x) becomes:

$$0.5(x)^{-0.5} - 1 = 0$$

$$(0.5)/(x)^{0.5} = 1$$

$$x = 0.25$$

This is the location along the X axis at which the ordinate, (y), is a maximum. The point on the X axis where the curve intersects the axis is obviously where (y) is equal to zero, and therefore, setting Eq. (55) equal to zero and solving for the value of (x) at the intersection:

$$(x)^{0.5} - x = 0$$

$$x = 0, \text{ or } 1.0$$

(x) = 0 occurs at the origin, while the other point at which (y) is zero is at a point one unit away from the origin. The maximum ordinate is 0.25 units away from the origin, or therefore 25% of the distance from the origin to the point of intersection.

The constants in Eq. (55) were assumed to be equal to unity, but a change in constants will not affect the location of the maximum ordinate, assuming the exponents to be kept the same. This is shown in Table VI, where the same exponents as in Eq. (55) are used, but the constants changed. In this Table, the term "chord" is used to designate the distance from the origin to the point of intersection of the curve with the X axis.

TABLE VI

Effect of Constants on Location of Maximum Ordinate

Equation	Location of Max. Ordinate (Units)	Chord (Units)	Location of Max. Ordinate (% Chord)
$y = (x)^{0.5} - 2(x)$	0.0625	0.25	25
$y = 2(x)^{0.5} - 2(x)$	0.2500	1.00	25
$y = (x)^{0.5} - 0.5(x)$	1.0000	4.00	25
$y = (x)^{0.5} - 0.2(x)$	6.2500	25.00	25

The computations necessary for completing Table VI were made in exactly the same manner as outlined for Eq. (55).

Table VII shows, however, that when the exponents are changed, the location of the maximum ordinate is also changed. It will be noted in Table VII that the exponents of the fourth and eighth, and the sixth and ninth, equations are the same, and therefore the locations of the maximum ordinate in percent of the chord are the same.

TABLE VII

Effect of Exponents on Location of Maximum Ordinate

Equation	Location of Max. Ordinate (Units)	Chord (Units)	Location of Max. Ordinate (% Chord)
$y = (x)^{0.5} - (x)^{1.5}$	0.3333	1.000	33.33
$y = (x)^{0.5} - (x)^{2}$	0.3968	1.000	39.68
$y = (x)^{0.5} - (x)^{2.5}$	0.4484	1.000	44.84
$y = (x)^{0.5} - (x)^{3}$	0.4883	1.000	48.83
$y = (x)^{0.5} - (x)^{3.5}$	0.5227	1.000	52.27
$y = (x)^{0.5} - (x)^{4}$	0.5521	1.000	55.21
$y = (x)^{0.5} - (x)^{10}$	0.7296	1.000	72.96
$y = 2(x)^{0.5} - 0.5(x)^{3}$	0.8503	1.741	48.83
$y = 3(x)^{0.5} - 0.0000006(x)^{4}$	45.2900	82.023	55.21

It should be remembered that if the constants of two or more equations are equal, the thickness ratios, (as defined in Section 2.40), are the same, regardless of whether or not the exponents are the same. Hence, in the first seven equations of Table VII, the thickness ratios are equal.

In the equations in Table VII, only the exponents of the second terms were changed. However, it is also true that the location of the maximum ordinate will vary if the exponents of the first terms are changed while the exponents of the second terms are unchanged. A general two term power curve equation will now be developed for this case where the value and location of the maximum ordinate , and the distance to the axis intersection, are given. The distance from the origin to the point where the curve intersects the axis will be called the "chord", and, if for the sake of consistency with the rest of the text the axis is assumed to be the X axis, then $(y) = 0$ at the intersection.

The maximum ordinate, or the maximum value of (y), occurs where the curve is parallel to the X axis, or, in other words, where the slope, (dy/dx), is equal to zero. Thus, utilizing these definitions, two equations can be written giving the relationship of the exponents to determine the location of the maximum ordinate along the chord. Using the general form of a two term equation from Eq. (31), then, and writing it for the point at the end of the chord where (y) is equal to zero, we have for the first equation:

$$y = k_1(x)^{n_1} - k_2(x)^{n_2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56)$$

The first derivative of Eq. (56) gives the slope, which is equal to zero when (y) is a maximum. Hence, for the second equation:

$$(dy/dx) = n_1k_1(x)^{n_1 - 1} - n_2k_2(x)^{n_2 - 1} = 0 \quad \ldots\ldots\ldots\ldots\ldots (57)$$

If (L), for length of the chord, is substituted for (x) in the first equation, and (CL) is substituted for (x) in the second equation, where (C) represents a constant whose value is the position of the maximum ordinate in hundredths of the chord, we get for Eq. (56):

$$k_1(L)^{n_1} = k_2(L)^{n_2}$$

or:

$$(k_1/k_2) = (L)^{n_2}/(L)^{n_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (58)$$

Eq. (57) becomes:

$$n_1k_1(CL)^{n_1 - 1} = n_2k_2(CL)^{n_2 - 1}$$

or:

$$(k_1/k_2) = n_2(CL)^{n_2 - 1}/n_1(CL)^{n_1 - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (59)$$

The relationship between the exponents can now be found by equating Eqs. (58) and (59), as follows:

$$\frac{(L)^{n_2}}{(L)^{n_1}} = \frac{n_2(CL)^{n_2-1}}{n_1(CL)^{n_1-1}}$$

$$\frac{n_1(C)^{n_1-1}(L)^{n_1-1}}{(L)^{n_1}} = \frac{n_2(C)^{n_2-1}(L)^{n_2-1}}{(L)^{n_2}}$$

$$n_1(C)^{n_1-1} = n_2(C)^{n_2-1}$$

or:

$$(n_1/n_2) = (C)^{n_2-n_1} \quad \text{..............................(60)}$$

Eq. (60) readily permits the determination of the location of the maximum ordinate for any two term power curve equation if the exponents are known. For example, if (n_1) and (n_2) are assumed to be 0.5 and 1.0, respectively, the location of the maximum ordinate is found as follows:

$$(0.5/1.0) = (C)^{1.0-0.5}$$

$$0.5 = (C)^{0.5}$$

$$\log (0.5) = 0.5 \log (C)$$

$$\log (C) = (0.69897-1)/(0.5)$$

$$= 1.39794-2$$

or :

$$C = N(0.39794-1)$$

$$= 0.25$$

Hence the maximum ordinate is located at $(x) = 0.25(c)$, or at 25% of the chord from the origin.

Further evaluating Eq. (60) for fixed values of (n_1) and various values of (n_2), in the same manner as in the above illustrative example, a series of curves can be computed and plotted from which (n_2) can be easily found for any

maximum ordinate location desired. This has been done in Fig. 17.

For demonstrating the method of developing two term power curve equations by the use of Fig. 17, a few numerical examples will be given. First, let it be assumed that the equation to be developed is to have a first term exponent of 0.4, and that the curve is to have a chord of 262 inches, a maximum ordinate of 26 inches, and the location of the maximum ordinate is to be at $(x) = 112$ inches.

Then the maximum ordinate is at:

$$C = (112/262)(L)$$

$$= 0.4275(L)$$

or:

$$= 42.75\% \text{ of the chord}$$

From Fig. 17, for $(n_1) = 0.4$ and $100(C) = 42.75$, the required value of (n_2) is:

$$n_2 = 2.625$$

The equation is now:

$$y = k_1(x)^{0.4} - k_2(x)^{2.625} \quad \text{.........................(61)}$$

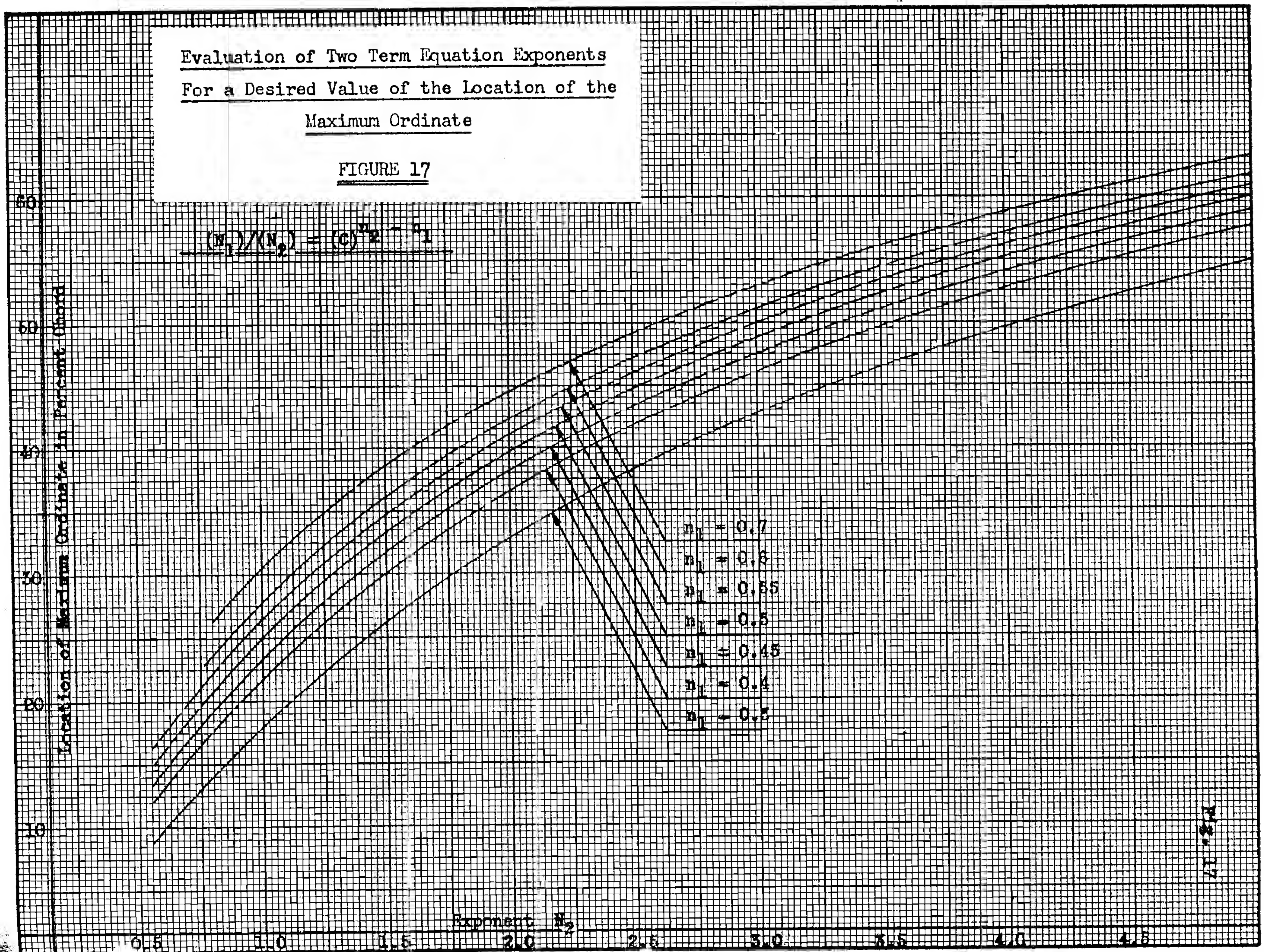
All that now remains is to solve for (k_1) and (k_2), which can be done in the manner outlined in Section 2.41. Hence, at the location of the maximum ordinate, $y = 26$ inches and $x = 112$ inches, Eq. (61) becomes:

$$y = k_1(112)^{0.4} - k_2(112)^{2.625} = 26 \quad \text{....................(62)}$$

The second equation is written for the point of intersection of the curve with the X axis, where $(y) = 0$ inches and $(x) = 262$ inches. Therefore:

$$y = k_1(262)^{0.4} - k_2(262)^{2.625} = 0 \quad \text{....................(63)}$$

Eqs. (62) and (63) become, after raising to the required powers and



Evaluation of Two Term Equation Exponents For a Desired Value of the Location of the Maximum Ordinate

FIGURE 17

Fig. 17

extracting the roots, (which, as mentioned previously, can usually be done most easily by the use of logarithms);

$$y = 6.602(k_1) - 239442(k_2) = 26$$

and:

$$y = 9.2752(k_1) - 2228632(k_2) = 0$$

Solving these equations simultaneously, (k_1) and (k_2) are found to be:

$$k_1 = 4.63834$$
$$= N(0.66636)$$

and:

$$k_2 = 0.000019304$$
$$= N(0.28565-5)$$

Eq. (61) can now be put in terms of the final equation as follows:

$$y = N(0.66636)(x)^{0.4} - N(0.28565-5)(x)^{2.625} \quad \text{..................(64)}$$

In a similar manner as shown above, the use of a value of $(n_1) = 0.5$ instead of 0.4, with the chord, maximum ordinate, and location of the maximum ordinate the same as in the first example, produces the following equation:

$$y = N(0.49636)(x)^{0.5} - N(0.14341-4)(x)^{2.3} \quad \text{..................(65)}$$

If (n_1) is taken as 0.6, and all other variables are again assumed the same as in the first example, the equation becomes:

$$y = N(0.33520)(x)^{0.6} - N(0.82865-4)(x)^{2.05} \quad \text{..................(66)}$$

Table VIII presents an evaluation of Eqs. (64), (65), and (66) for various assumed values of (x), and the values from this Table are plotted on Fig. 18.

TABLE VIII

Evaluation of Eqs. (64), (65), and (66)

(x)	(y)		
	Eq.(64)	Eq.(65)	Eq.(66)
1	4.64	3.14	2.11
2	6.12	4.43	3.30
5	8.83	7.01	5.67
10	11.64	9.89	8.54
25	16.72	15.45	14.43
60	22.96	22.58	22.26
112	26.00	26.00	26.00
162	23.32	23.11	22.99
204	16.61	16.24	16.01
238	7.97	7.68	7.51
262	0.00	0.00	0.00

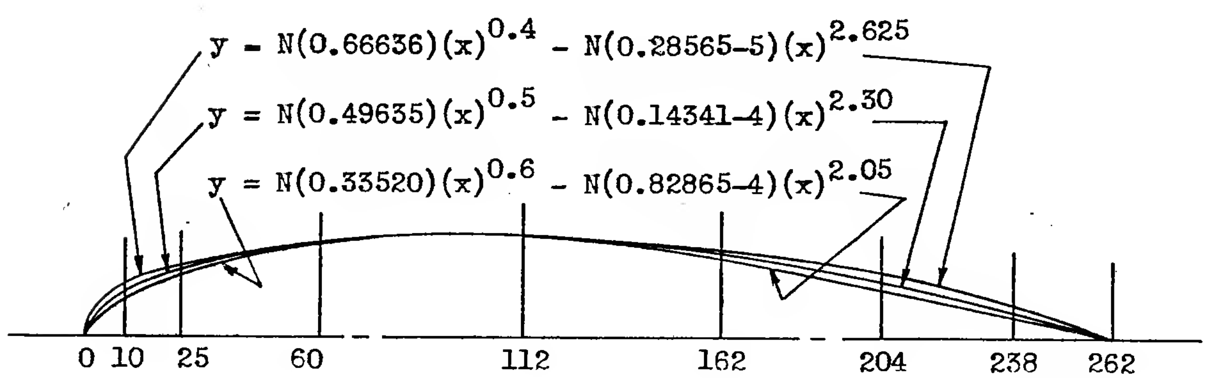


FIGURE 18

By comparing the curves of the three equations, it becomes at once apparent that an equation that has a large exponent for the first term gives a curve with narrow, or more pointed, front and aft sections, and that the bluntness increases as the exponent of the first term decreases. In all these equations the first term could be placed after the second, thus making the second term the first, but the equation would still be the same. In this work, however, it shall be understood that by the first term is meant that term which has the lowest exponent of (x), which, in most cases, is less than unity, and that the term number is designated in the order of the increase of the exponents of (x).

The illustrative examples presented above show the means whereby two term power curve equations can be developed when the maximum ordinate conditions, and the chord, are specified. It has also been demonstrated how the shape of the curve can be modified by the choice of the exponents, and by the values chosen for the maximum ordinate and its location.

2.43 - Other Conditions for which Two Term Equations Can be Developed:- It has been shown in Sections 2.41 and 2.42, respectively, the manner in which two term power curve equations can be written when (1) the exponents are given, or assumed, and (2) the magnitude and location of the maximum ordinate, and the chord, are given. In the latter case, the first term exponent has to be assumed.

Should it be required to develop the equation for a curve passing through two points, in addition to the origin, two possible methods are available. If the ordinate of the second point is larger than that of the first, and the slope is of no importance, a single term equation can be found, as shown in Fig. 13 and Eqs. (49), (50), and (51). If, however, a certain slope is specified at one or both points, or if the ordinate of the second point is

less than that of the first point, then the "trial and error" method outlined in Section 2.60 must be used to find the desired equation.

2.50 - Conversion from Two to Three or More Term Equations;- In the previous sections, the development of two term equations was demonstrated. However, it may sometimes be found that, after a two term equation has been developed, and the ordinates computed and plotted, the resulting curve may not quite meet the requirements, and some slight modification is needed. This may be done by changing the exponents of one or both of the two terms, but such a method can involve several trial and error assumptions before the desired curve is obtained.

A simpler method consists of adding a third term, the value of which can usually be found quickly. This method should not be confused with that of originally developing a three term equation for the curve, as will be outlined later on. Instead, the third term is added solely as a means of modifying a fundamental two term equation.

Assume that the finally desired curve has already been drawn, in preliminary form - either sketched or faired with splines or french curves, or known from some previous design. The curve developed by the two term power curve equation, (which we have assumed needs modification), can then be placed over this preliminary curve, the points of origin made to coincide, and the X axis raised or lowered until the two term curve most closely coincides with the desired curve. The X axis of the two term curve will then form an angle with the X axis of the other. The tangent of this angle then becomes the constant of a third term in (x), the exponent of which is unity. If the X axis of the two term curve has to be lowered the third term will be negative, and positive if the X axis is raised.

Consider, for example, the curve represented by Eq. (64). Let it be assumed that, in order to make this curve coincide more closely with a desired curve, it is necessary to raise the X axis by 0.5 inches at the Station (x) = 100 inches. The constant of the third term will then be + 0.005, or, in log form, N(0.69697-2). Hence, Eq. (64) would be modified to:

$$y = N(0.66636)(x)^{0.4} + N(0.69897-2)(x) - N(0.28565-5)(x)^{2.625} \quad ....(67)$$

Changing the equation in this fashion, instead of initially developing a three or more term equation, will sometimes save time and computations. However, when using this method, the "chord", or distance from the origin to the point of intersection with the axis, will increase depending upon the amount the axis was inclined. To find the length of the new chord it will be necessary to compute, and plot, a few stations near where the point of intersection with the axis is expected. If this new chord is not materially different from the original one, but it is still necessary to maintain exactly the original chord, then the last term should be changed, or an additional term added.

2.60 - Development of Multi-Term Power Curve Equations:- In Section 2.30, it was demonstrated how a single term power curve equation could be developed for a curve starting at the origin and passing through two given points, or through one point at which the slope was known. (See Figs. 13 and 14). However, when using the single term form, the curve could not be made to become parallel to, or intersect, the axis at any time, but instead continued ever further away from both X and Y axes as (x) was increased.

To make the curve of a single term equation become parallel to the X axis, and eventually intersect it, a second term can be derived and subtracted from the first, making a two term power curve equation as discussed in Sections

2.40 through 2.43. The two term power curve equation can only be developed when certain conditions are specified, however, and hence is somewhat limited in its use, although not nearly as much as the single term equation.

Therefore, in order to derive a curve which can meet more specified conditions, or can be more generally used, a multi-term equation consisting of three or more terms must be employed. The development of such an equation may require several trial and error assumptions, but the final result will be a smooth continuous curve, with uniformly varying curvature, and passing through the desired points. The number of trial and error processes required to develop the equation will also be found to be substantially reduced after a little experience has been gained in the use of the method.

To illustrate the method used in this process, consider Fig. 19, where it is assumed that a multi-term equation has to be found for a curve passing through points (P) and (Q), and some other point which need not be yet given. It will also be assumed that the required slope at point (Q) is given. The process will only be shown here for the first two points, as a numerical example later on will illustrate the entire development of a multi-term equation.

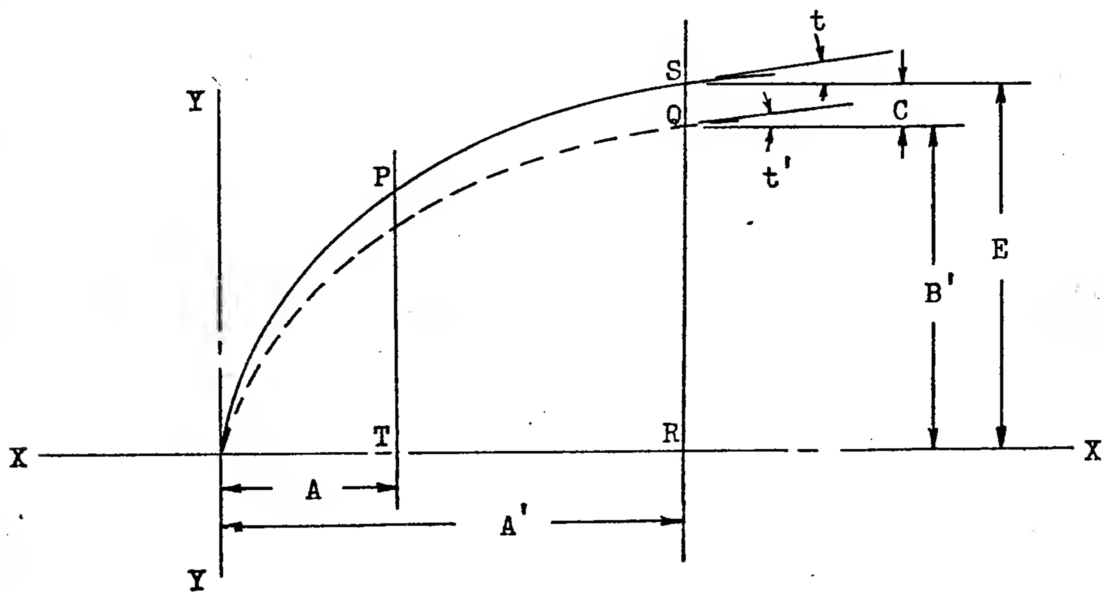


FIGURE 19

A single term equation can easily be found for the solid curve in Fig. 19 passing through point (P), since, by assuming the exponent, (n), the constant (k) can be found from Eq. (53) and the equation written in the form:

$$y = k(x)^n$$

If the required ordinate and abscissa of point (Q) are B' and A', respectively, substitution of this value of A' in the equation for the curve passing through point (P) will give the ordinate of this curve at Station R, or at the same abscissa as point (Q). Let this ordinate be called (E). The ordinate desired at point (Q) is B', however, and therefore the second term of the equation will have to reduce the ordinate (E) by the amount (E - B'), or (C').

Similarly, since the required slope at point (Q) is given, we can substitute the value of A' in the derivative of the first term and find the slope the solid curve will have at Station R. If this slope at point (Q) which is required is called (t'), and the slope of the solid curve at Station R is designated (t), the second term of the equation will have to change the slope by an amount (t - t'), which we shall call (D').

Writing these equations:

$$y = k_1(A')^{n_1} - C' = B' \quad \text{.........................(68)}$$

or, since $E = k_1(A')^{n_1}$, we get:

$$C' = E - B' \quad \text{.............................(69)}$$

and:

$$(dy/dx) = n_1k_1(A')^{n_1-1} - D' = t' \quad \text{...................(70)}$$

or, since $t = n_1k_1(A')^{n_1-1}$:

$$D' = t - t' \quad \text{.............................(71)}$$

The second term can then be considered as the equation of a curve which, at an abscissa of (A'), will have an ordinate of (C') and a slope of (D'). Hence, from Eqs. (53) and (54), the required constant and exponent, respectively, of the second term are:

$$k_2 = C'/(A')^{n_2} \quad \text{.............................(72)}$$

and:

$$n_2 = (A'D')/(C') \quad \text{.............................(73)}$$

The equation has now been developed to the extent:

$$y = k_1(x)^{n_1} - k_2(x)^{n_2}$$

In the above equation, (k_1), (k_2), and (n_2) are known, and (n_1) is also known since it was assumed at first from a consideration of the shape of curve desired at the origin. (See Figs. 15 and 16).

In evaluating the equation at this point, it will usually be found that the ordinate at point (P) has been reduced by the second term, if this is negative, or increased if the second term is positive. Should such be the case, then a re-development of the equation becomes necessary with a correction to the value of the ordinate used in determining the constant and exponent of the first term.

For the third term of the equation, the procedure to be used is the same as just outlined for finding the second term, providing the slope and coordinates of the third point are known. If one of these is unknown, it must be assumed in order to permit the slope and ordinate corrections to be found. Here again, after the first value of the third term has been determined, the entire equation should be evaluated to ascertain the effect of this third term on the ordinates. If the effect is too large, the entire process has to be repeated with corrections made to the first two terms, and a new third term

found.

It should be remembered that by employing these trial and error corrections, and developing new equations to finally make the curve fit the specified points, the position of the maximum ordinate might shift somewhat. In many cases, this shift will be so slight as to be of no importance. If the position of the maximum ordinate must remain fixed, however, then perhaps another exponent can be assumed for the first term, or the abscissa and/or the ordinate of the first or second point can be changed somewhat without affecting to any degree the overall requirements for the final complete curve.

So far it has been assumed in most cases that negative terms are used after the first term. In many cases, though, positive terms are needed as for example when reverse curvature is desired. The signs of the terms are usually determined automatically in the development, however, for if (C) turns out to be negative when a negative term is assumed, the term should be positive, and vice versa. In addition, the signs can almost always be determined by visual inspection.

In order to obtain a smooth continuous curve to pass through a given set of points, every attempt should be made to obtain exponents successively increasing in magnitude, such as 0.5 for the first term, from 1.0 to 2.0 for the second term, from 3.0 to 5.0 for the third, and so on. If the equation developed for a given set of points does not have exponents increasing in such an order, it indicates that the specified ordinates of one or more of the points are too large or small to fit on a smooth curve. Hence, the ordinate or ordinates in question should be modified slightly and a new equation developed. It will almost always be found that such modifications can be made readily, as the ordinate of each point is usually specified as a maximum or minimum,

and therefore the correction can be made in the opposite direction.

It will be noted that in the discussion on Fig. 19, the slope at point (Q) was assumed to be known. If the ordinate at point (Q) is greater than at point (P), however, and the slope is not specified at point (Q), a single term can be used for the curve passing through these two points. In this case, the exponent would be found by Eq. (51), and the constant determined by substituting this exponent back in an equation similar to Eq. (49) or (50). Thus, by developing another term for the last point on the curve, and combining this term with the single term for the first two points, a two term equation would be obtained for the entire curve, instead of the three term equation obtained in the example discussed.

The reader should carefully note the above statement that the use of a single term for the first two points only applies when the ordinate of the second point is larger than that of the first. This is because a single term equation, as noted previously, can not be made to define a curve which at some point becomes parallel to the X axis and then reverses its slope. Hence, should the ordinate of the second point be smaller than that of the first point, and the slope of the second point is unknown, then this slope must be assumed and two terms found, as before, for the first two points. (Eqs. (68) through (73)).

An illustrative example is given below to show how the equation can be developed for a curve passing through three points. Suppose that the curve is to pass through the three points, (P), (Q), and (R) as indicated on Fig. 20, and that the slope at point (Q) is to be zero, and the slope at point (R) is to be -0.22.

An exponent of 0.5 will be assumed for the first term. Hence, for point (P):

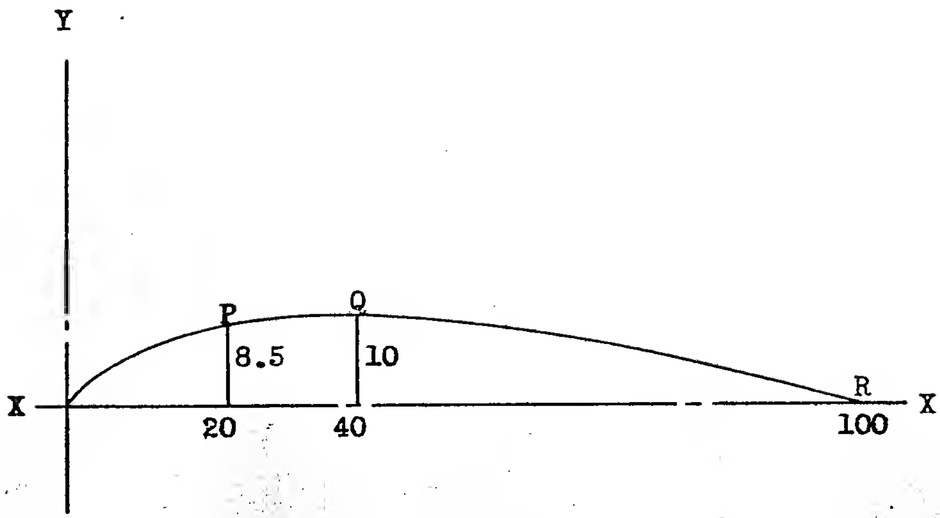


FIGURE 20

$$A = 20$$
$$B = 8.5$$

Since we know that the second term will have to be negative in order to make the curve intersect the X axis, it will reduce the value of (B) at point (P). Therefore, assuming for the first term that (B) = 9.0, (k) can be found from Eq. (53):

$$k = B/(A)^n$$
$$= 9.0/(20)^{0.5}$$

or, in logs:

$$k = \log(9) - (0.5)\log(20)$$
$$= N(0.95424 - 0.65052)$$
$$= N(0.30372)$$

Therefore the first term is:

$$y = N(0.30372)(x)^{0.5} \qquad (74)$$

The value of (y) when (x) = (A') = 40 is:

$$y = N(0.30372)(40)^{0.5}$$
$$= N(0.30372 + 0.5 \log 40)$$

$$= N(0.30372 + 0.80103)$$

$$= N(1.10475)$$

$$= 12.728$$

The correction which has to be applied to the ordinate by the second term is then, from Eq. (69):

$$C' = E - B'$$

$$= (12.728 - 10)$$

$$= 2.728$$

The slope of the curve defined by Eq. (74) at an abscissa $(x) = (A') = 40$ is found from the first derivative:

$$(dy/dx) = (0.5)N(0.30372)(40)^{-0.5}$$

$$= N(\log 0.5 + 0.30372 - 0.5 \log 40)$$

$$= N((0.69897-1) + 0.30372 - 0.80103)$$

$$= N(0.20166-1)$$

$$= 0.15910$$

The slope correction to be applied by the second term is found from Eq. (71), where, in this case, (t'), the required slope at point (Q), is zero:

$$D' = t - t'$$

$$= (0.15910 - 0)$$

$$= 0.15910$$

The exponent of the second term is now found from Eq. (73):

$$n_2 = (A'D')/C'$$

$$= (40)(0.15910)/(2.728)$$

$$= 2.3328$$

and the constant is found from Eq. (72):

$$k_2 = C'/(A')^{n_2}$$

$$= 2.728/(40)^{2.3328}$$

or, in logs:

$$k_2 = N(\log 2.728 - 2.3328 \log 40)$$

$$= N(0.43584 - 3.73733)$$

$$= N(0.69851-4)$$

Since C' and D' are positive, the second term must be negative, and the equation so far becomes:

$$y = N(0.30372)(x)^{0.5} - N(0.69851-4)(x)^{2.3328} \qquad \text{.............. (75)}$$

The third point has the ordinate $A'' = 100$, $B'' = 0$, and the slope is -0.22. To determine the ordinate correction to be supplied by the third term, the two term equation as given by Eq. (75) must first be evaluated for $(x) = A'' = 100$:

$$y = N(0.30372)(100)^{0.5} - N(0.69851-4)(100)^{2.3328}$$

$$= N(0.30372 + 0.5 \log 100) - N(0.69851-4)(2.3328 \log 100)$$

$$= N(1.30372) - N(1.36417)$$

$$= 20.124 - 23.129$$

$$= -3.005 \qquad \text{.. (75a)}$$

The minus sign indicates that the curve defined by Eq. (75) would lie 3.005 units below the X axis at $(x) = A'' = 100$, and therefore the third term should be positive in order to correct this condition. Hence, from Eq. (69), the correction to be applied to the ordinate by the third term is:

$$C'' = E - B''$$

$$= (-3.005) - 0$$

$$= -3.005 \qquad \text{.................................. (75b)}$$

The slope correction to be applied by the third term is found by first determining the slope Eq. (75) would have at $(x) = A'' = 100$. Therefore, differentiating Eq. (75), we have:

$$(dy/dx) = (0.5)N(0.30372)(x)^{-0.5} - (2.3328)N(0.69851-4)(x)^{1.3328}$$

For $(x) = A'' = 100$, this equation has the value:

$$(dy/dx) = N\left[\log(0.5) + (0.30372) - 0.5 \log (100)\right] - N\left[\log (2.3328) + (0.69851-4) + 1.3328 \log (100)\right]$$

$$= N(0.00269-1) - N(3.73205-4)$$

$$= 0.10062 - 0.53958$$

$$= - 0.43896 \quad \ldots\ldots\ldots\ldots\ldots\ldots (75c)$$

Here again the sign is negative, showing that the third term must be positive to raise the slope from −0.43896 to the specified −0.22. Hence, from Eq. (71):

$$D'' = t - t''$$

$$= (-0.43896) - (-0.22)$$

$$= - 0.21896 \quad \ldots\ldots\ldots\ldots\ldots\ldots (75d)$$

The exponent of the third term is now found from Eq. (73):

$$n_3 = A''D''/C''$$

$$= (100)(0.21896)/(3.005)$$

$$= 7.28650$$

The constant is found from Eq. (72):

$$k_3 = C''/(A'')^{n_3}$$

$$= (3.005)/(100)^{7.28650}$$

$$= N\left[(\log 3.005) - (7.28650 \log 100)\right]$$

$$= N(0.47784 - 14.573)$$

$$= N(0.90484-15)$$

Note that in determining the values of (n_3) and (k_3) above, the negative signs for C'' and D'' were not used, since they are only used to determine the sign of the constant (k_3), which has been shown to be required to be positive.

The equation of the curve is now found by adding the third term to Eq. (75):

$$y = N(0.30372)(x)^{0.5} - N(0.69851-4)(x)^{2.3328} + N(0.90484-15)(x)^{7.28650} \quad \ldots(76)$$

In computing the ordinates from Eq. (76), it will be found that at point P the ordinate is 8.459 instead of the specified 8.5. The difference is 0.041, which, in most cases, is negligible. Should a closer result be needed, however, the equation should be re-developed starting with an ordinate at point P of perhaps 9.1 instead of the value of 9.0 assumed in this example.

2.61 - Determination of Terms when Slopes are Unknown: - In the illustrative example given in Section 2.60, it was assumed that the slopes at points Q and R, (Fig. 20), were known. Very often, however, the slopes are not known, and the equation must be developed in another manner.

As mentioned previously, if the ordinate at point Q is larger than that at point P, and the slope is unknown, a single term can be found to fit these two points by the use of Eqs. (51) and (49) or (50). To then find the second term, (for the third point), the slope at the third point must be assumed, and the procedure followed as given in the illustrative example in Section 2.60.

Another method by which the equation may be developed for three points, when all the slopes are unknown, is to first find a single term for the first point by either assuming a slope at this point or the exponent of the first

term, Eqs. (53) and (54). Then this single term equation can be evaluated up to the abscissa of the third point, and a second term found to take into account the last two points, as demonstrated below.

Consider the illustrative example shown by Fig 20 in Section 2.60. Let it be assumed that, in addition to passing through points P and Q, the curve must also pass through the points $(x) = 90$, $(y) = 2.5$ and $(x) = 100$, $(y) = 0$, (point R). Here we have a case where four points must be satisfied, but it will be noted that the slopes at the last two points are not specified. This example is then indicative of the condition mentioned above where a term is needed to take into account the last two points on the required curve.

Eq. (75) gives the equation as developed for the first two points, P and Q:

$$y = N(0.30372)(x)^{0.5} - N(0.69851\text{-}4)(x)^{2.3328} \quad \text{.................... (75)}$$

This equation must now be evaluated at the abscissae at the last two points, $(x) = 90$ and $(x) = 100$. Hence, for $(x) = 90$, the value of (y) for Eq. (75) is:

$$\begin{aligned} y &= N(0.30372 + 0.5 \log 90) - N(0.69851\text{-}4 + 2.3328 \log 90) \\ &= N(0.30372 + 0.97712) - N(0.69851\text{-}4 + 4.55891) \\ &= 19.091 - 18.089 \\ &= 1.002 \end{aligned}$$

At station $(x) = 100$, the (y) for Eq. (75) is:

$$\begin{aligned} y &= N(0.30372 + 0.5 \log 100) - N(0.69851\text{-}4 + 2.3328 \log 100) \\ &= N(1.30372) - N(5.36417\text{-}4) \\ &= 20.124 - 23.129 \\ &= -3.005 \end{aligned}$$

The ordinate correction at station $(x) = 90$ is, from Eq. (69):

$$C'' = E - B''$$
$$= 1.002 - 2.50$$
$$= -1.498$$

and, at station $(x) = 100$, where $(y) = E = 0$:

$$C''' = E - B'''$$
$$= (-3.005) - 0$$
$$= -3.005$$

Both of these corrections have negative signs, showing that the third term must be positive. Hence, at station $(x) = 90$, the required constant of the third term is, from Eq. (72):

$$k_3 = C''/(A'')^{n_3}$$
$$= (1.498)/(90)^{n_3} \qquad \text{.........................(77)}$$

and, at station $(x) = 100$:

$$k_3 = C''/(A'')^{n_3}$$
$$= 3.005/(100)^{n_3} \qquad \text{.........................(78)}$$

The value of (n_3) required to satisfy both Eqs. (77) and (78) can now be found by equating the two:

$$(1.498)/(90)^{n_3} = (3.005)/(100)^{n_3}$$
$$(100/90)^{n_3} = (3.005/1.498)$$
$$(1.111)^{n_3} = 2.007$$
$$n_3 = \log(2.007)/\log(1.111)$$
$$= 6.61695$$

The value of (k_3) is now found by substituting back in either Eq. (77) or (78), as follows:

$$k_3 = 3.005/(100)^{6.61695}$$

$$= N\left[\log(3.005) - 6.61695 \log(100)\right]$$

$$= N(0.47785 - 13.23390)$$

$$= N(0.24395\text{-}13)$$

The third term is then:

$$y = N(0.24395\text{-}13)(x)^{6.61695}$$

and the final equation is Eq. (75) plus this third term, or:

$$y = N(0.30372)(x)^{0.5} - N(0.69851\text{-}4)(x)^{2.3528} + N(0.24395\text{-}13)(x)^{6.61695} \quad \ldots\ldots(79)$$

The above shows the manner in which equations can be developed without the slopes being known, or the method by which two points can be used to permit the development of a single term in the equation.

By employing the principles evolved in this and the preceeding section, multi-term power curve equations can be developed for a curve passing through many points. Obviously, the greater the number of points specified through which the curve must pass, the more complicated the equation becomes, involving more and more computations. Therefore, it is advisable to keep to a minimum the number of points required to lie on the curve. It will usually be found that this can be done in many instances, for although it may be initially specified that all the given points should lie on the curve, many of these points will be of secondary importance, or will be maximum or minimum values where slight variations can be permitted.

2.70 - Correction of Equations for Change in Thickness Ratio:- The ratio of the external dimensions of thickness and length of a body are of considerable use and interest where any kind of fluid flow is involved. This ratio is known as "thickness ratio", and is defined as the maximum ordinate of the body, or curve, divided by the length. This is usually expressed in percent of the length. Thus, for example, if the maximum ordinate is 12.5 units and the length 100 units, the thickness ratio would be 12.5/100, or 0.125, or 12.5%.

It is very often desired, after the equation has been developed for a desired curve, to change the thickness ratio for one reason or another. This can be most easily done, without resorting to the necessity for developing another equation from the beginning.

Consider, for example, Eq. (64) in Section 2.42, where the maximum ordinate was specified as 26 inches and the length, or chord, as 262 inches:

$$y = N(0.66636)(x)^{0.4} - N(0.28565\text{-}5)(x)^{2.625} \quad \ldots\ldots\ldots\ldots\ldots\ldots(64)$$

Suppose that it is desired to change the maximum ordinate to 25 inches, and a new equation is therefore needed. It has been previously noted that the constants of an equation control the thickness ratio, and that the exponents have nothing to do with it. Therefore, if the thickness ratio is varied, only the constants have to be changed, and this occurs in the same proportion.

Hence, to change from a thickness of 26 inches to 25 inches in the above example, without changing the chord, it is only necessary to multiply the constants by the ratio 25/26, or, using logs, the correction to be added to each constant becomes:

$$= \log(25) - \log(26)$$

$$= (2.39794\text{-}1) - (1.41497)$$

$$= 0.98297\text{-}1$$

Adding this amount to the log of each constant in Eq. (64), the corrected equation becomes:

$$y = N\left[(0.66636) + (0.98297\text{-}1)\right](x)^{0.4} - N\left[(0.28565\text{-}5) + (0.98297\text{-}1)\right](x)^{2.625}$$

$$= N(0.64933)(x)^{0.4} - N(0.26862\text{-}5)(x)^{2.625} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(80)$$

Should it be desired to thicken a curve, the same procedure is used, the multiplier of the constants then being greater than unity. In the above

example, a two term equation was used for convenience, but the method is applicable to equations with any number of terms, provided, of course, that all the constants are corrected in the same ratio.

To change the thickness ratio by lengthening or shortening the chord, while leaving the value of the maximum ordinate unchanged, the procedure becomes slightly more elaborate. In this case, all the (x)'s have to be multiplied by a number equal to the ratio of the chord on hand to the desired chord. Since these (x)'s are raised to a power, the multiplier must also be raised to the same power. The multiplier, however, being a constant, can be separated and raised to the desired power and then can be included in the constant of the term.

Considering Eq. (64) again, assume that it is desired to change the chord from 262 inches to 270 inches, leaving the maximum ordinate at 26 inches as in the original curve. Then each (x) must be multiplied by the inverse ratio of the change in chords, or 262/270. Using logs, the multiplier becomes:

$$= \log (262) - \log (270)$$

$$= (3.41830-1) - (2.43136)$$

$$= 0.98694-1$$

Substituting this value in Eq. (64), we get:

$$y = N\left[(0.66636) + 0.4(0.98694-1)\right](x)^{0.4} - N\left[(0.28565-5) + 2.625(0.98694-1)\right](x)^{2.625}$$

To demonstrate the method of multiplying logarithms by fractional constants, the above equation will be evaluated in detail:

$$0.4(0.98694-1) = (0.39478-0.4)$$

The anti-log of the number on the right hand side of the above equation can not be found unless the negative number is a whole digit, such as 1, 2, 3, etc. Hence, 0.6 can be added to both numbers in order to make the negative number equal to -1, or:

$$= (0.39478 + 0.6) - (0.4 + 0.6)$$

$$= 0.99478-1$$

This logarithm is now in a form to be added to, or subtracted from, other logarithms to permit the final anti-log to be found in log tables.

In a similar manner:

$$2.625(0.98694-1) = (2.24070-2.625)$$

Adding 0.375 to both numbers to make the negative number a whole digit:

$$= (2.24070 + 0.375) - (2.625 + 0.375)$$

$$= 2.61570-3$$

These values must now be added to the respective constants in Eq. (64), as shown below:

The constant for $(x)^{0.4}$ becomes:

$$= N\left[(0.66636) + (0.99478-1)\right] = N(0.66114)$$

and the constant for $(x)^{2.625}$ becomes:

$$= N\left[(0.28565-5) + (0.61570-1)\right] = N(0.90135-6)$$

Eq. (64), corrected to a chord length of 270 inches and a maximum ordinate, as originally, of 26 inches, then is:

$$y = N(0.66114)(x)^{0.4} - N(0.90135-6)(x)^{2.625} \quad \text{..................(81)}$$

Note that, by this method of including the multiplier in the constants, the exponents are unchanged, thus keeping to a minimum the number of computations required to compute the ordinates of the new curve.

CHAPTER III

CURVATURE AND POINTS OF INFLECTION

3.00 - Points of Inflection on a Curve:- It has been previously stated that the curves defined by one term power equations start at the origin, turn swiftly in one direction or the other, (depending on the value of the exponent), and then continue their course with the ordinate continuing to increase as the abscissa is increased. The direction of curvature is always the same, or, in other words, the curve continues to bend more and more in the same direction for as far as it is extended. Such a curve is known as a " continuous" curve, for, although it will never bend enough to become exactly parallel to either axis, it will continue to bend infinitesimal amounts as it approaches infinity.

When a negative term is added to a single term equation with an exponent less than unity, the tendency is to make the curve bend around some more. If the exponent of the second term is larger than that of the first term, the curve will eventually become parallel to the axis and then intersect it later on. This curve also does not change its direction of curvature, and therefore is continuous.

An easy way to understand the meaning of a "continuous" curve is to visualize segments of the curve being drawn by a compass. As long as the segments can be drawn with the compass center on the same side of the curve, the curve is continuous. When the compass has to be shifted to the other side of the curve in order to draw the next segment, the curve becomes "discontinuous", or reverses.

A reverse curve is usually obtained from a power curve equation, with a first term exponent of less than unity, when a positive term is added, provid-

-ed that the exponent of this positive term is large enough. Obviously, if the positive term has a small exponent, or constant, and the previous negative term, or terms, has a large exponent, the effect of the positive term will just be to slightly decrease the rate of curvature without affecting the continuity of the curve. Hence, it can not generally be said that all positive terms will cause a curve to reverse, but instead the mathematical equations of the curve must be examined to determine if such a condition exists.

It may also happen that a negative term will cause a curve to reverse. For example, referring to Fig. 21, Curve C represents a power curve equation of two terms, the first being positive with an exponent larger than unity, with the second term being negative.

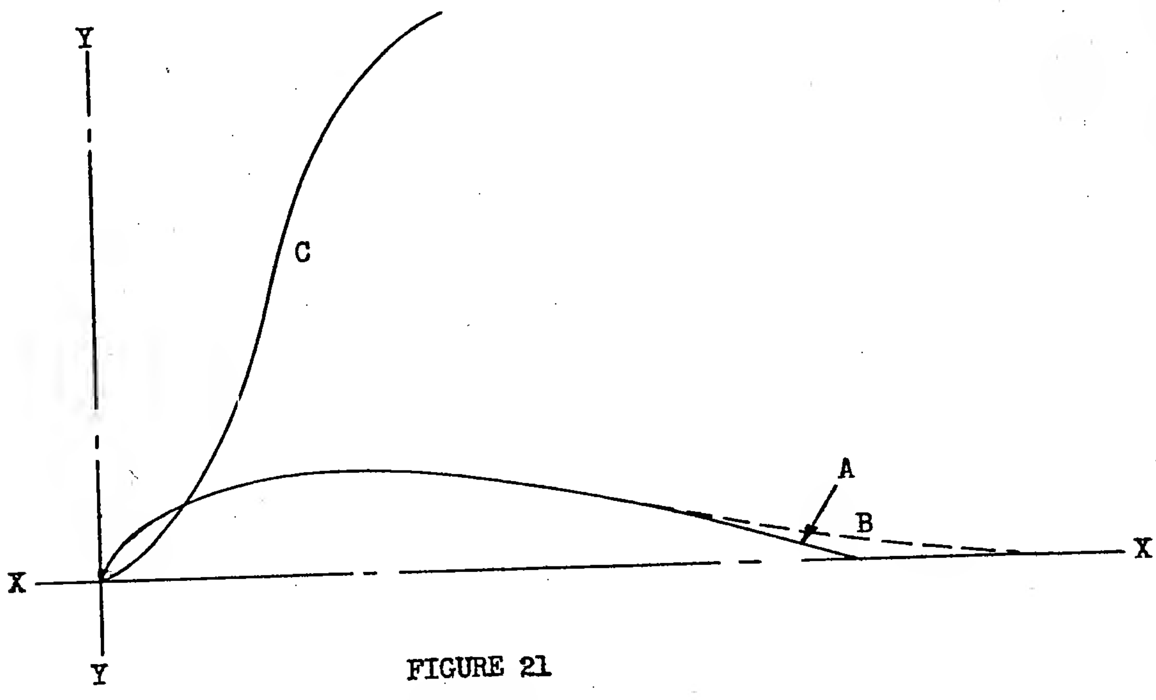


FIGURE 21

Here the negative term causes the curve to reverse, since the first term exponent was larger than unity. Curve A represents a continuous curve which could be formed from a two term equation with a first term exponent less than unity and a negative second term. If a third term should be added to this two term equation, and the term be positive with a large exponent, Curve B might result, with a reverse in it.

Many multi-term equations may have reverses in their curves so small as to be hardly noticeable by eye, and therefore it is necessary to mathematically examine every equation developed to determine if any reverse in curvature is present. Reversals in curvature can be very detrimental to preserving a smooth fluid flow over a surface, and a breakdown of a smooth flow means an unnecessary drag, or resistance, for the object. Therefore, in streamlining work where fluid flow exists, and the resistance of the object to this flow must be kept to a minimum, reversals in curvature should be eliminated if at all possible, except sometimes near the end of the chord.

To determine if any reverse curvature exists, the equation of the curve must be searched for "points of inflection", a "point of inflection" being defined as the point where the curvature changes its sign, or reverses. It will be remembered that the first derivative of an equation gives the equation of the slope of the curve at any point. When this equation is set equal to zero, and the value, or values, of the independent variable solved for, these values represent the points at which the slope is zero. The second derivative of the original equation, or the derivative of the first derivative equation, is a function of the curvature, and, when set equal to zero, gives the value, or values, of the independent variable at which the curvature is zero. If the curvature changes sign after this point, this is a point of inflection as the curvature has reversed.

The easiest means of finding whether such points exist is to plot the second derivative equation. Where the curve intersects the X axis the curvature obviously changes sign, and hence a point of inflection exists. Consider, for example, Fig. 20 and Eq. (76), where the specified slope at the end of the chord, (point R), was -0.22. The first derivative of Eq. (76) is:

$$dy/dx = (0.5)N(0.30372)(x)^{-0.5} - (2.3328)N(0.69851\text{-}4)(x)^{1.3328} +$$

$$(7.2865)N(0.90484\text{-}15)(x)^{6.28650}$$

$$= N\left[(\log 0.5) + (0.30372)\right](x)^{-0.5} - N\left[(\log 2.3328) + (0.69851\text{-}4)\right](x)^{1.3328}$$

$$+ N\left[(\log 7.2865) + (0.90484\text{-}15)\right](x)^{6.28650}$$

$$= N(0.00269)(x)^{-0.5} - N(0.06639\text{-}3)(x)^{1.3328} + N(0.76736\text{-}14)(x)^{6.2865} \quad .....(82)$$

The second derivative is obtained in the same manner:

$$d^2y/dx^2 = -(0.5)N(0.00269)(x)^{-1.5} - (1.3328)N(0.06639\text{-}3)(x)^{0.3328} +$$

$$(6.2865)N(0.76736\text{-}14)(x)^{5.2865}$$

$$= -N(0.70166\text{-}1)(x)^{-1.5} - N(0.19116\text{-}3)(x)^{0.3328} + N(0.56577\text{-}13)(x)^{5.2865} \quad .....(83)$$

Eqs. (76), (82), and (83) have been evaluated for varying values of (x), and the results are given in Table IX and plotted on Fig. 22.

TABLE IX

Evaluation of Eqs. (76), (82), and (83)

x	y	dy/dx	d^2y/dx^2
1	2.012	1.0050	-0.50466
2	2.841	0.7086	-0.17983
5	4.479	0.4400	-0.04765
10	6.257	0.2931	-0.01925
30	9.629	0.0753	-0.00764
50	9.658	-0.0691	-0.00679
70	6.995	-0.1918	-0.00516
90	2.397	-0.2499	+0.00035
100	0.000	-0.2250	+0.00608

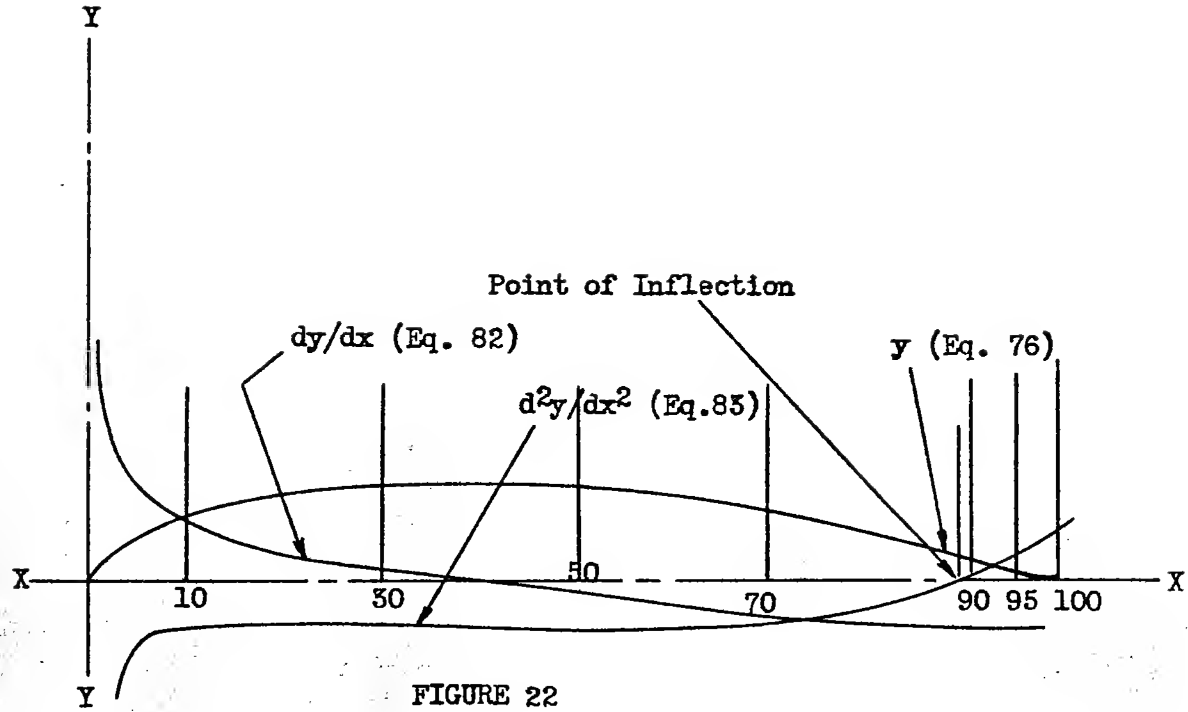


FIGURE 22

Fig. 22 shows that the curve of the second derivative intersects the X axis at approximately (x) = 88 inches, indicating a point of inflection, or a point where the curve reverses. As a matter of interest, the point at which the slope of the original curve is zero was specified at (x) = 40 inches, (see Fig. 20), and this is shown on Fig. 22 where the curve of the first derivative intersects the axis. This is the maximum ordinate position.

Let us now assume that it is desired to change the equation so that the curve does not reverse at any point. This can be done by a trial and error method, either by assuming the slope at the end point or by assuming the exponent of the last term. As a first approximation, therefore, let the slope at the end point, (x) = 100, be changed from the original value of -0.16 to -0.22. Only the third term need be changed, and the required corrections to be applied by the third term to the ordinate and slope at (x) = 100 can be found as originally done in developing Eq. (76).

The ordinate correction remains unchanged from Eq. (75b), since no change has been made. Hence:

$$C'' = -3.005 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(75b)$$

The slope which the first two terms of the equation would have at (x) = 100 remains -0.43896, from Eq. (75c), but the slope correction is now, from Eqs. (71) and (75c):

$$D'' = -0.43896 - (-0.16)$$
$$= -0.27896$$

The new exponent is then, from Eq. (73):

$$n_3 = A''D''/C''$$
$$= (100)(0.27896)/3.005$$
$$= 9.28320$$

and the new constant is, from Eq. (72):

$$k_3 = C''/(A'')^{n_3}$$
$$= 3.005/(100)^{9.28320}$$
$$= N(0.91144-19)$$

Eq. (76) is now, modified for a slope of -0.16 at (x) = 100 inches:

$$y = N(0.30372)(x)^{0.5} - N(0.69851-4)(x)^{2.3328} + N(0.91144-19)(x)^{9.28320} \ldots\ldots(84)$$

Checking the effect that this new third term will have on the maximum ordinate at (x) = 40 inches, we substitute this value of (x) in Eq. (84) and find that the maximum ordinate is 10.0006 inches instead of the originally specified 10.00 inches. This effect is negligible and need not be further considered.

To find the first and second derivatives of Eq. (84), only the third term need be investigated, as the first two terms will be unchanged from Eqs. (82) and (83), respectively. The derivation will not be carried out again, as

the procedure is the same as shown in deriving Eqs. (82) and (83). Therefore:

$$dy/dx = N(0.00269)(x)^{0.5} - N(0.06639\text{-}3)(x)^{1.3328} + N(0.87914\text{-}18)(x)^{8.2832} \quad ...(85)$$

and:

$$d^2y/dx^2 = -N(0.70166\text{-}1)(x)^{-1.5} - N(0.19116\text{-}3)(x)^{0.3328} + N(0.79734\text{-}17)(x)^{7.2832} \quad .....(86)$$

Eqs. (84), (85), and (86) have been evaluated in Table X below, and plotted on Fig. 23.

TABLE X

Evaluation of Eqs. (84), (85), and (86)

x	y	dy/dx	d^2y/dx^2
1	2.012	1.0050	-0.50466
2	2.841	0.7086	-0.17983
5	4.479	0.4400	-0.04765
10	6.257	0.2931	-0.01925
30	9.629	0.0753	-0.00788
50	9.644	-0.0710	-0.00699
70	6.882	-0.2006	-0.00554
90	2.131	-0.2463	+0.00319
100	0.000	-0.1600	+0.01542

Fig. 23 shows that, by changing the slope of the curve at station (x) = 100 inches from -0.22 to -0.16, the point at which a reverse occurs in the curve of the equation has shifted further forward than when the slope was -0.22 at the end point. This shift could have been seen beforehand, as obviously a decrease in the slope at the end point would move the position of reversal forward, but the example has been worked out to show the trial and error manner

in which the problem must be solved.

To eliminate the reversal in the curve, therefore, the slope at station (x) = 100 inches must be increased in a positive direction. For a second assumption in this case, then, the exponent of the last term will be assumed instead of the slope in order to demonstrate how this method may be used.

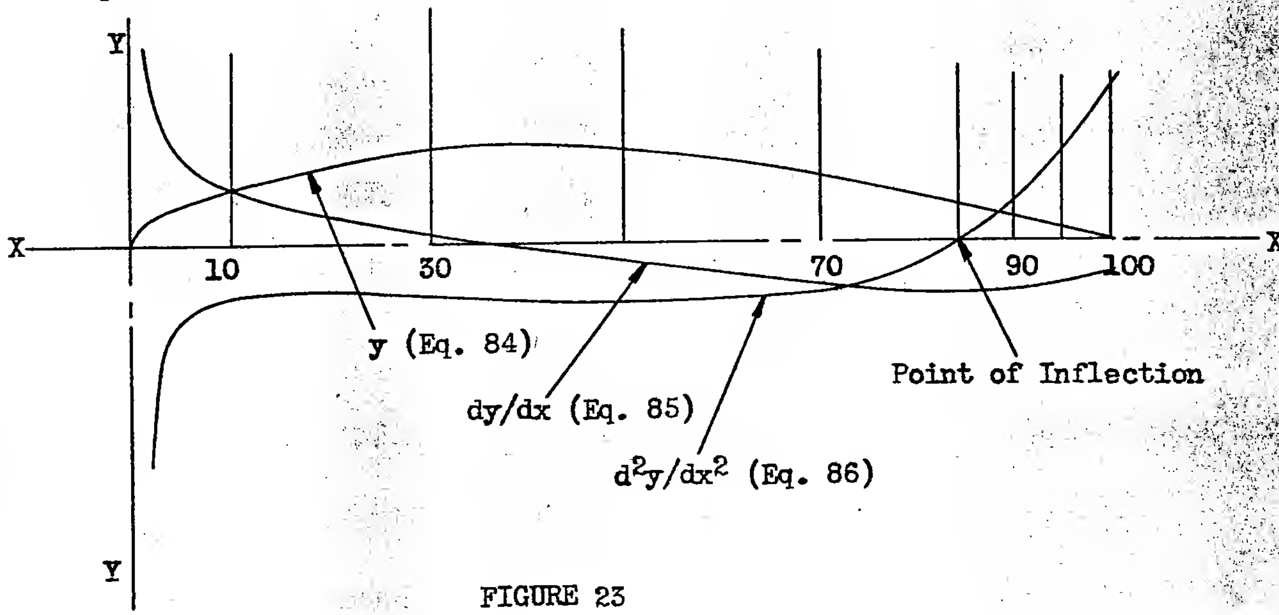


FIGURE 23

the value of (d^2y/dx^2) for the first two terms of Eq. (83), at station (x) = 100 inches, is:

$$d^2y/dx^2 = -N(0.70166-4) - N(0.85682-3)$$

$$= -0.007692$$

In order for (d^2y/dx^2) for the complete equation to be zero at station (x) = 100, the (d^2y/dx^2) of the third term must be -0.007692. Assume now that the third term exponent is 5.435. Then, since the ordinate correction required of the third term is still -3.005, (Eq. 75b), the value of k_3 is found from Eq. (72):

$$k_3 = C''/(A'')^{n_3}$$

$$= 3.005/(100)^{5.435}$$

$$= N(0.60784-11)$$

and the third term is:

$$y = N(0.60784-11)(x)^{5.435} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(87)$$

Hence:

$$dy/dx = N(0.34304-10)(x)^{4.435} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(88)$$

and:

$$d^2y/dx^2 = N(0.98993-10)(x)^{3.435} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(89)$$

Evaluating Eq. (89) for (x) = 100, we get:

$$d^2y/dx^2 = N(0.85993-10)$$

$$= 0.007243$$

This does not yet quite check the required value of 0.007692, but for this example will be taken as a close enough check. The third term of the equation as given by Eq. (87) will, however, eliminate any reversal even beyond station (x) = 100 inches, and can be therefore considered satisfactory. The final equation is now:

$$y = N(0.30372)(x)^{0.5} - N(0.69851-4)(x)^{2.3328} + N(0.60784-11)(x)^{5.435} \ldots\ldots(90)$$

The first and second derivatives of Eq. (90) are:

$$dy/dx = N(0.00269)(x)^{-0.5} - N(0.06639-3)(x)^{1.3328} + N(0.34304-10)(x)^{4.435} \ldots(91)$$

and:

$$d^2y/dx^2 = -N(0.70166-1)(x)^{-1.5} - N(0.19116-3)(x)^{0.3328} + N(0.98993-10)(x)^{3.435} \ldots(92)$$

Eqs. (90), (91), and (92) are eveluated below in Table XI, and plotted on Fig. 24.

Fig. 24 clearly shows no point of inflection until station (x) = 100, and therefore no curve reversal. Eq. (90) then represents the equation of a

TABLE XI

Evaluation of Eqs. (90), (91), and (92)

x	y	dy/dx	d^2y/dx^2
1	2.012	1.0050	-0.50466
2	2.841	0.7086	-0.17983
5	4.479	0.4400	-0.04765
10	6.259	0.2931	-0.01925
30	9.633	0.0761	-0.00776
50	9.708	-0.0643	-0.00647
70	7.204	-0.1815	-0.00512
90	2.697	-0.2605	-0.00250
100	0.000	-0.2757	-0.00045

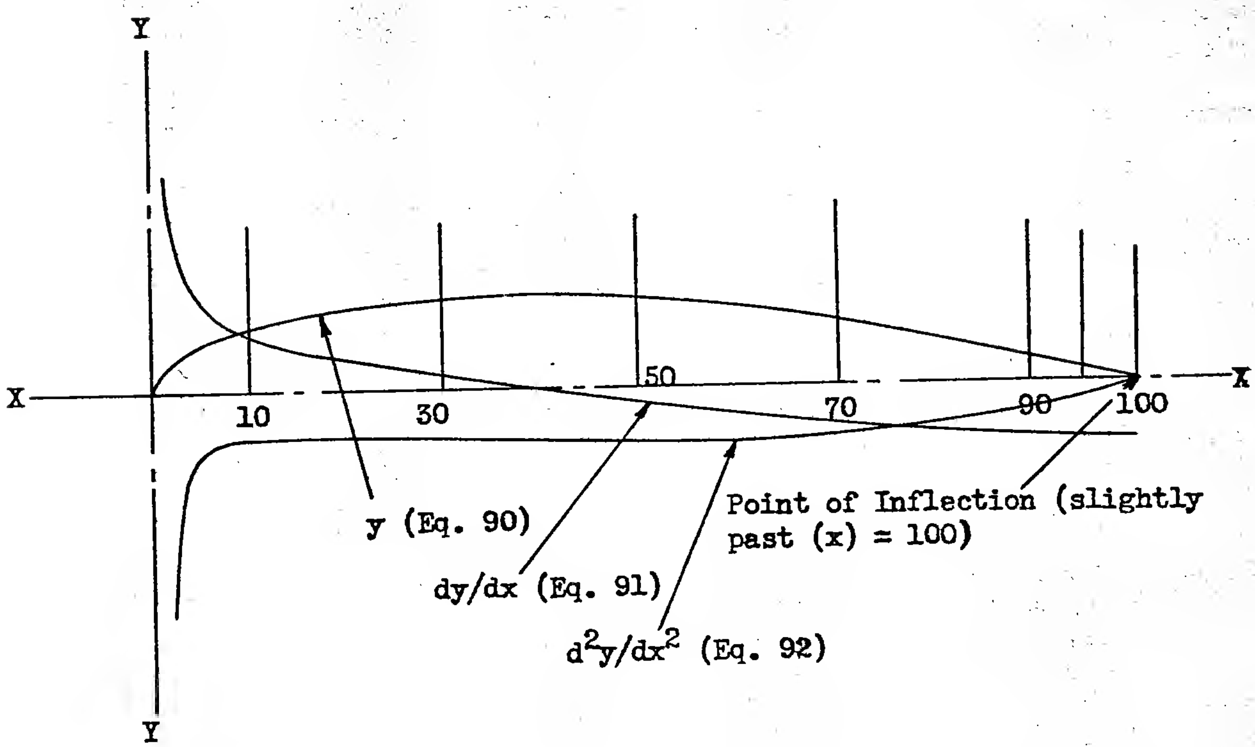


FIGURE 24

continuous curve, with no reversals, passing through the points (x) = 20, (y) = 8.5 and (x) = 40, (y) = 10, with the maximum ordinate at station (x) = 40. These were the conditions specified for Fig. 20 and Eq. (76), except that it was found necessary to change the slope at the end point, (station R in Fig. 20), in order to eliminate a reversal in the curve. This results in the new third term given in Eq. (90).

3.10 - Determining the Smoothness of a Curve: - It has been shown in Section 3.00 how the curve of the second derivative is used to check for points of reversal in the basic equation. Such reversals, as has been pointed out before, are usually very detrimental to the preservation of an undisturbed fluid flow over the surface, and can be the cause of excessive resistance of the object when moving through a fluid. In aircraft work, where the fluid speeds are continually increasing as aircraft speeds increase, the drag caused by reversals in the contour lines of fuselages, nacelles, etc., may be many times magnified.

It is usually not sufficient, however, to only examine a curve for points of reversal, as abrupt changes in slope, or curvature, of the contour lines may have the same effect as reversals in curvature. It has long been known that the pressure distribution over a body is a function of the degree of curvature of the body, and, in turn, that this pressure distribution governs the magnitude of the local velocity of the fluid moving past the body. Rapid increases or decreases in curvature of the body cause rapid accelerations, or decelerations, in the local fluid velocities, resulting in a breakdown of the smooth flow, with corresponding high drags.

It is therefore necessary, particularly where high fluid velocities prevail, to insure that the curvature of the body changes as gradually as possible. This can be checked by the use of the same second derivative curve used in examining for points of reversal, for this curve really represents the

rate of change of slope of the basic curve, and is therefore essentially a function of its curvature. Hence, a plot of the (d^2y/dx^2) curve should form a smooth continuous line, with no peaks, or sudden changes, or waviness. Obviously this curve should also lie on one side of the axis, for to cross the axis would signify a reversal in the basic curve.

For example, referring to the (d^2y/dx^2) curve on Fig. 23, it can be seen that this curve forms a smooth line up until shortly before it crosses the X axis at the point of reversal in the basic curve. The curvature of the basic line can therefore be considered to change very little from about station (x) = 10 to station (x) - 70, indicating a good smooth basic line. Ahead of station (x) = 10 the curvature changes very rapidly, which, in this illustrative case of an airfoil section, can not be helped since a rounded, rather than a pointed, front section was desired for other reasons. Aft of station (x) = 70, the change of curvature is quite large, and, even if no reversal were present in the basic curve at this point, this rapid change in curvature might prove to be quite harmful to the preservation of a smooth undisturbed fluid flow.

Fig. 24, however, shows that the change of curvature aft of station (x) = 70 was substantially reduced when the reversal in the basic curve was eliminated. Here the (d^2y/dx^2) curve is a very smooth flat line, showing that, aft of station (x) = 10, as before, the curvature of the basic line hardly changes. This means that the fluid, in flowing around the contour line, will at no place encounter sudden increases or decreases in curvature, and therefore should not suddenly break down and cause undue drag increases.

All lines developed, therefore, should be carefully checked for the possibility of sudden curvature changes, particularly where high fluid velocities are involved. If such changes in curvature occur, a study of the first and

second derivative curves will show where corrections should be made to the power curve equation. Obviously, no set rules can be stated for applying these corrections, but a little experience in such matters will enable the engineer to accomplish it in a short time.

3.20 - Curvature and Radius of Curvature:- In Section 3.10 it was stated that the curve of the second derivative, (d^2y/dx^2), can be used for checking the curvature of the basic line. This is quite true, but mathematically, the value of (d^2y/dx^2) does not give the actual curvature, but instead is a direct function of the curvature.

Very often, however, it may be desired to know the actual curvature, or the radius of curvature. The latter is very useful in permitting the line to be reproduced, as it is sometimes easier to work with radii than with a table of ordinates. The expressions for curvature, K, and radius of curvature, R, are therefore given below. The derivations of these equations will not be outlined here, as they may be found in any good reference to the Calculus:

$$K = (d^2y/dx^2)/\left[1 + (dy/dx)^2\right]^{1.5} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(93)$$

and:

$$R = 1/K = \left[1 + (dy/dx)^2\right]^{1.5}/(d^2y/dx^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(94)$$

As an illustrative example in the use of the expressions for curvature and radius of curvature, Eqs. (84) and (90) are evaluated below in Tables XII and XIII, respectively. The calculations are self-explanatory, the values of (dy/dx) and (d^2y/dx^2) for Eq. (84) being taken from Table X, and the corresponding values for Eq. (90) from Table XI.

The values from Tables XII and XIII are plotted on Figs. 25 and 26, Fig. 25 showing the curvature, K, and Fig. 26 the radius of curvature, R.

TABLE XII

Curvature and Radius of Curvature for Eq. (84)

x	y	dy/dx	d^2y/dx^2	K	R
1	2.012	1.0050	-0.50466	-0.17709	-5.64
2	2.841	0.7086	-0.17983	-0.09769	-10.22
5	4.479	0.4400	-0.04765	-0.03654	-27.40
10	6.259	0.2931	-0.01925	-0.01701	-58.70
30	9.629	0.0753	-0.00788	-0.00781	-128.00
50	9.644	-0.0710	-0.00699	-0.00693	-144.30
70	6.882	-0.2006	-0.00554	-0.00522	-191.50
90	2.131	-0.2463	+0.00319	+0.00293	341.50
100	0.000	-0.1600	+0.01542	+0.01480	67.50

TABLE XIII

Curvature and Radius of Curvature for Eq. (90)

x	y	dy/dx	d^2y/dx^2	K	R
1	2.012	1.0050	-0.50466	-0.17709	-5.64
2	2.841	0.7086	-0.17983	-0.09769	-10.22
5	4.479	0.4400	-0.04765	-0.03654	-27.40
10	6.257	0.2931	-0.01925	-0.01701	-58.70
30	9.633	0.0761	-0.00776	-0.00769	-130.00
50	9.708	-0.0643	-0.00647	-0.00642	-155.90
70	7.204	-0.1815	-0.00512	-0.00488	-205.00
90	2.697	-0.2605	-0.00250	-0.00233	-430.00
100	0.000	-0.2757	-0.00045	-0.00040	-2500.00

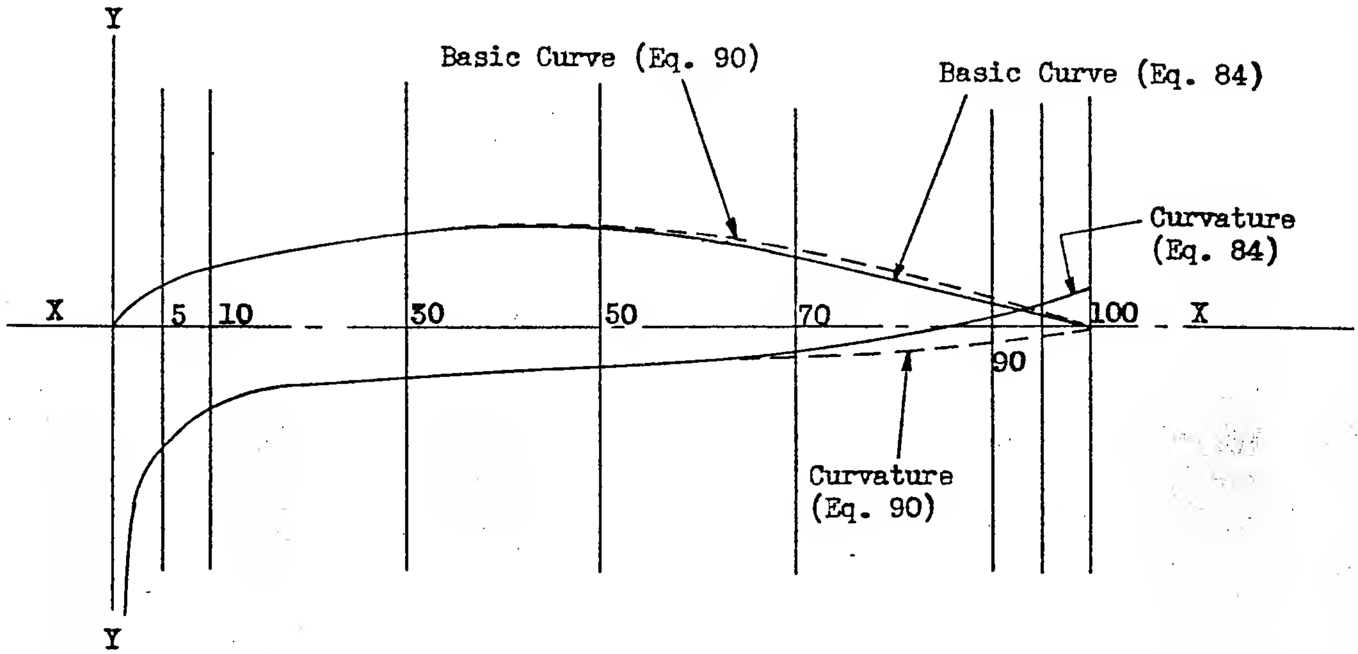


FIGURE 25

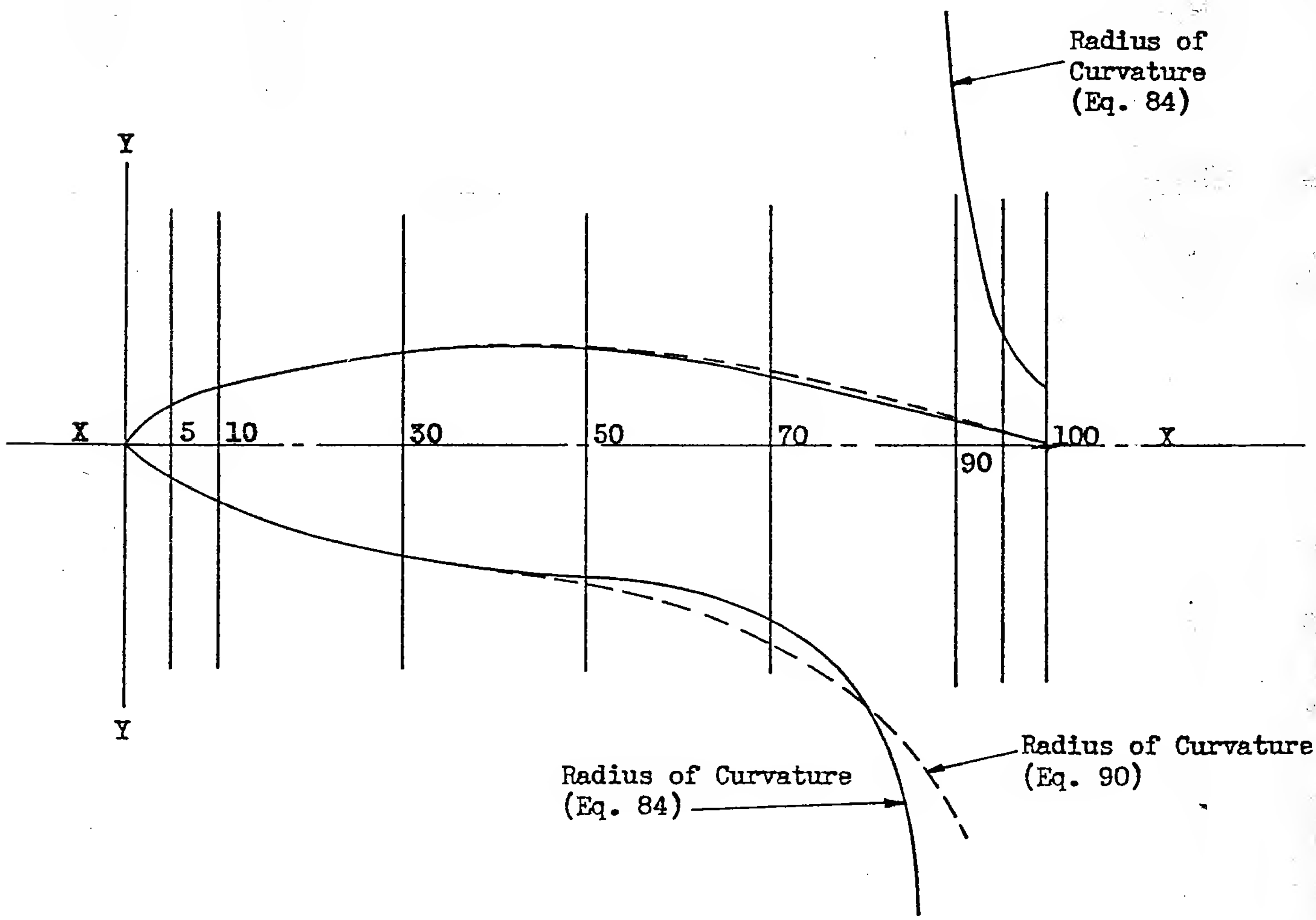


FIGURE 26

An inspection of Fig. 25 shows that the lines of curvature for Eqs. (84) and (90) are substantially the same as the (d^2y/dx^2) curves of Figs. 23 and 24, respectively. This indicates why the (d^2y/dx^2) curve can be used in most cases for checking the smoothness of the basic curve. However, when the values of (dy/dx) are large, or vary considerably, it is advisable to use the actual values of curvature, and not (d^2y/dx^2), in checking the basic curve.

Fig. 26 clearly shows the reversal in curvature in Eq. (84), as it will be noted that the radius has shifted from one side of the X axis to the other.

Further uses for the values of curvature and radius of curvature will be outlined in later sections.

CHAPTER IV

FURTHER APPLICATIONS OF THE POWER CURVES

The previous chapters have shown the methods whereby power curve equations can be developed to obtain various curves and shapes meeting certain requirements. In addition, several numerical examples have been given during the course of these developments illustrating how the equations are used in these cases.

In this chapter, further examples will be given to show the application of the power curve equations. Quite obviously it is not possible to cover all of the conditions under which the equations can be used, but a thorough study of the examples already presented, and those to be presented in this chapter, will enable the reader to develop his own equations for any condition desired.

4.00 - <u>Combining a Power Curve with a Circle or an Ellipse</u>;- Power curves can be combined with a circle or an ellipse in a similar manner as an ellipse can be combined with a circle or another ellipse, as shown in Sections 2.21 and 2.22. For example, assume that it is required to combine a power curve with the circle shown in Fig. 27, and that the power curve is to have its origin at point O and

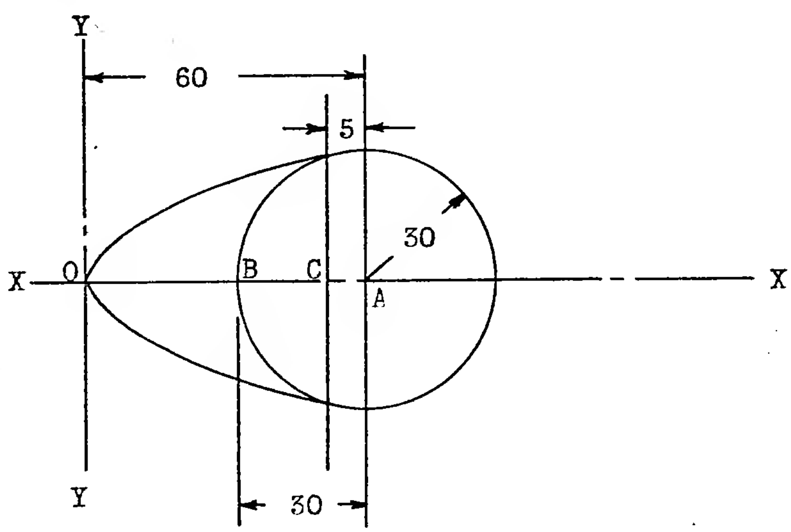


FIGURE 27

be tangent to the circle at station C, 55 units from point O.

In order to determine the values of the exponent and constant for the power curve equation, the ordinates and slopes at the point, or points, through which the curve is to pass must first be found. Hence, since the curve is to be tangent to the circle at station C, the ordinate and slope at station C must be found.

The ordinate may be found from either Eq. (2) or Eq. (7a), depending upon where one chooses to call the origin of the circle. Using Eq. (7a), the value of (y) at station C is:

$$y = \sqrt{2rx - x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7a)$$

$$= \sqrt{(2)(30)(25) - (25)^2}$$

$$= 29.580$$

and, from Eq. (9), the slope at staion C is:

$$dy/dx = (r - x)/(y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

$$= (30 - 25)/29.580$$

$$= 0.16903$$

Developing the power curve equation for $A = 55$, $B = 29.580$, and $t = 0.16903$, and using Eqs. (53) and (54):

$$n = (55)(0.16903)/29.580$$

$$= 0.3143$$

and:

$$k = 29.580/(55)^{0.3143}$$

$$\log(k) = \log(29.580) - 0.3143 \log(55)$$

$$= 1.47100 - 0.54700$$

$$= 0.92400$$

or: $\quad k = N(0.92400)$

The required power curve equation is then:

$$y = N(0.92400)(x)^{0.3143}$$

Now it may happen that the curve of the equation so developed is too blunt, or too sharp, or is unsatisfactory for some other reason. If such is the case, then a second point should be specified, say at a station 20 units from point O, through which the curve must pass, and a two term equation found by the trial and error method outlined in Sections 2.60 and 2.61. If this is still not satisfactory, additional points should be specified, and a three, or four, or more, term equation developed by the same trial and error method.

Consider, as a second example, the case shown in Fig. 28, where it is required to combine a power curve with an ellipse, the major axis of the ellipse being vertical. This problem can be solved by rotating the sketch so that the

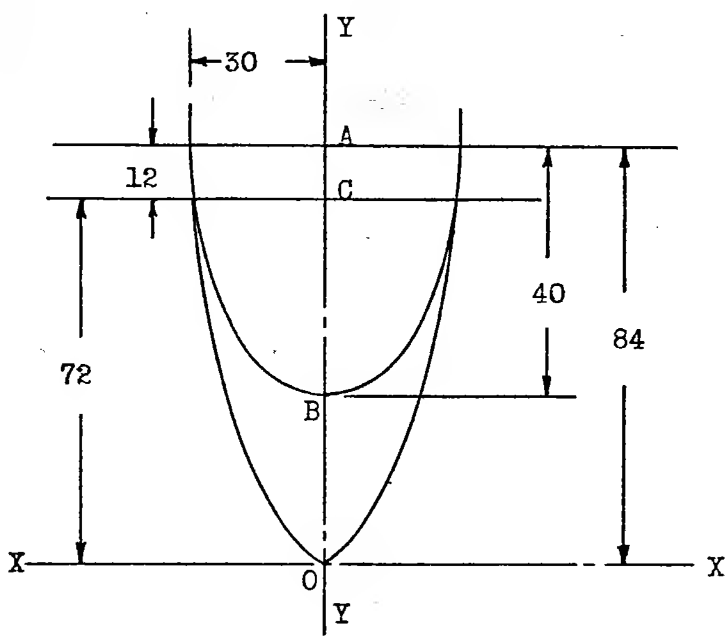


FIGURE 28

major axis coincides with the X axis, thus being consistent with all the sketches illustrated so far, but actually it makes little difference to leave the sketch in the form of Fig. 28, as will be seen in the following development.

Suppose that it is required to join a power curve with the given ellipse at station C. Here again, the slope and the abscissa at station C have to be found first.

Eq. (33) gives the power curve form of the equation of the ellipse, the major axis being coincident with the X axis and the origin at point B in Fig. 28. When the major axis coincides with the Y axis, it is only necessary to interchange (x) and (y), or:

$$x = (k_1y - k_2y^2)^{\frac{1}{2}} \qquad \text{.............................(33)}$$

where, from Eqs. (32a) and (32b):

$$k_1 = (2b^2)/a \qquad \text{.............................(32a)}$$

and:

$$k_2 = (b^2)/(a^2) \qquad \text{.............................(32b)}$$

Hence:

$$k_1 = (2)(30)^2/40$$

$$= 45.0$$

and:

$$k_2 = (30)^2/(40)^2$$

$$= 0.5625$$

At station C, (y) = 28, and therefore, from Eq. (33):

$$x = \sqrt{(45)(28) - (0.5625)(28)^2}$$

$$= 28.618$$

From Eq. (34), the slope at station C may be found, but first (x) and (y) must be interchanged since the axes have been interchanged:

$$dx/dy = (k_1 - 2k_2 y)/(2x) \qquad \text{.......................(34)}$$

$$= \frac{(45) - (2)(0.5625)(28)}{(2)(28.618)}$$

$$= 0.2534$$

The power curve equation may now be developed by again interchanging (x) and (y), and writing the equation in the form:

$$x = k(y)^n$$

If such were done, the exponent, (n), would be less than unity. However, the equation can just as well be developed in the familiar form, $y = k(x)^n$, the exponent simply becoming larger than unity to give the vertical shape desired. (See Fig. 15). To do this, the slope, (dy/dx), is needed, and this is simply:

$$dy/dx = 1/(dx/dy)$$

$$= 1/0.2534$$

$$= 3.9463$$

Therefore, for t = 3.9463, B = 72, and A = 28.618, Eqs. (53) and (54) give:

$$n = (28.618)(3.9463)/(72)$$

$$= 1.5686$$

$$k = (72)/(28.618)^{1.5686}$$

$$= N\left[\log (72) - 1.5686 \log(28.618)\right]$$

$$= N(0.58275-1)$$

The required equation is then:

$$y = N(0.58275-1)(x)^{1.5686}$$

In the combining of a power curve with an ellipse or a circle, or any other shape, to become tangent at any point, there is a limit to the extent to which this can be carried without encountering a reversal in the curves. This limit is that the origin of the power curve must lie within the straight line extensions of the tangent lines of the points of junction.

4.10 – Supplementary Curves: – It often happens that it is required to develop a curve to join two other curves, or that a given curve should be extended in such a manner that, for example, one section of the extension should be a certain shape, such as an ellipse or circle, and the other section another shape, such as could only be defined by a power curve. For example, consider Fig. 29, in which it is assumed that the curve to the right of station 200 is to be extended to the left in such a manner that the section from station 0 to station 40 is elliptical, with the ordinate at station 40 equal to 28. Between stations 40 and 200 another curve is to be developed which shall be tangent at

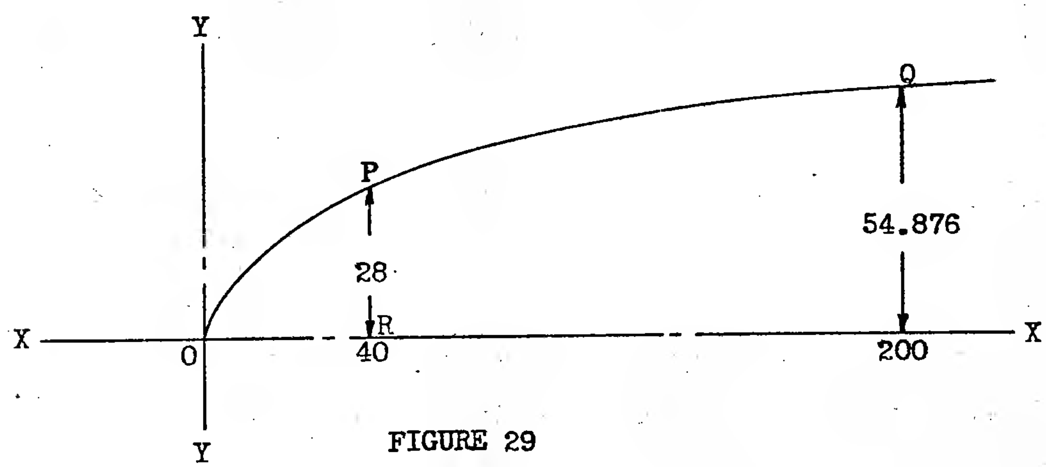


FIGURE 29

station 40 to the ellipse and at station 200 to the original curve. The slope at station 200 is 0.09136, and the ordinate 54.876.

It is not possible, in this case, to develop the usual power curve equation, (with its origin at point 0), for the entire curve to station 200,

since it was specified that from $(x) = 0$ to $(x) = 40$ the curve should be an ellipse. Neither is it possible to develop the usual form of power curve equation with its origin at point P, since the slope at the origin of a power curve equation in the form $y = k(x)^n$ is infinite. In this case, the slope at point P of the curve from $(x) = 40$ to $(x) = 200$ must be the same as the slope of the ellipse at this point.

Hence, the equation for the curve from $(x) = 40$ to $(x) = 200$ must be developed in another manner. This can be done by first assuming the slope at point P. This slope can not be less than the slope formed by a line drawn through points P and Q, and must be smaller than the slope formed by a line through points P and O. In the first case, the slope is:

$$dy/dx = (54.876 - 28)/(160)$$
$$= 0.1695$$

and, in the second case:

$$dy/dx = 28/40$$
$$= 0.70$$

Therefore, let the slope at point P be assumed as 0.30. Now if a straight line be drawn through point P with this slope, and the origin taken at point R, its equation is:

$$y = 28.0 + 0.30(x) \qquad \text{.............................(95)}$$

and, at station 200, (or $(x) = 160$ from point R, the origin for this section of the curve), the value of (y) would be:

$$y = 28.0 + 0.30(160)$$
$$= 76.0$$

This is larger than the required value of 54.876, and therefore a

correction must be applied in the manner demonstrated in Section 2.60. It will be noted that this development is exactly the same as outlined in Section 2.60, except that here we start by applying corrections to a straight line in order to curve it back to a specified ordinate, whereas in Section 2.60 the corrections were applied to an already curved line whose equation was in the form $y = k(x)^n$.

From Eq. (69), then, the correction to the ordinate at $(x) = 160$ is:

$$C' = 76 - 54.876$$
$$= 21.124$$

To find the slope correction necessary, the first derivative of Eq. (95) must be found and then evaluated at $(x) = 160$:

$$dy/dx = 0 + 0.30$$
$$= 0.30$$

The slope is obviously constant, since Eq. (95) represents a straight line, and the slope correction required at point Q is, from Eq. (71):

$$D' = 0.30 - 0.09136$$
$$= 0.20864$$

Since C' and D' are positive, the required correction term is negative. From Eqs. (72) and (73) the required constant and exponent, respectively, are now found:

$$n = (160)(0.20864)/21.124$$
$$= 1.5803$$

and:

$$k = (21.124)/(160)^{1.5803}$$

$$= N\left[\log(21.124) - 1.5803 \log(160)\right]$$

$$= N(4.33477\text{–}3 \quad - \quad 3.48317)$$

$$= N(0.85160\text{–}3)$$

The equation for the curve from point P to point Q is then given by:

$$y = 28.0 + \ 0.30(x) - N(0.85160\text{–}3)(x)^{1.5803} \quad \text{.................}(96)$$

It should be remembered that Eq. (96) assumes the origin at point R, and this must be allowed for in plotting the curve.

For the elliptical part of the curve from point O to point P, all that is known is one point on the ellipse, where (x) = 40, (y) = 28, and (dy/dx) = 0.30. It is therefore necessary to define the ellipse in terms of its semi-axes, (a) and (b). These values can be easily found, however, by the simultaneous solution of Eqs. (25) and (26):

$$y = (b/a)\sqrt{2ax - x^2} \quad \text{.........................}(25)$$

$$dy/dx = b^2(a - x)/a^2y \quad \text{.........................}(26)$$

Substituting the values at point P in Eq. (25), we have:

$$y = 28 = (b/a)\sqrt{2a(40) - (40)^2}$$

or:

$$b/a = 28/\sqrt{2a(40) - (40)^2} \quad \text{........................}(97)$$

and substituting in Eq. (26):

$$dy/dx = 0.3 = \frac{(b^2)(a - 40)}{(a^2)28}$$

or:

$$b/a = \frac{\sqrt{0.30}}{\sqrt{(a - 40)/(28)}} \quad \text{........................}(98)$$

These two equations for (b/a) are now equated in order to solve for (a):

$$(28)/\sqrt{2a(40) - (40)^2} = \sqrt{(0.3)} / \sqrt{(a - 40)/(28)}$$

$$(28)^2((a - 40)/28) = (0.3)\left[2a(40) - (40)^2\right]$$

$$a = 160$$

Putting this value of (a) back in Eq. (97), or Eq. (98), the value of (b) is found to be:

$$b = (160)(28)/\sqrt{(2)(160)(40) - (40)^2}$$

$$= 42.332$$

The equation for the ellipse from point O to point P is then, from Eq. (25):

$$y = (42.332/160)\sqrt{320(x) - x^2} \quad \text{.......................(99)}$$

Thus we have now three equations: the ellipse from station 0 to station 40, the supplementary curve from station 40 to station 200, and from there on the original curve. The curvatures of each curve should be next computed and plotted, and a check made whether or not they match closely, and are continuous. If not, the slope or odinate at station 40 should be changed, and new equations developed until the desired curve continuity is obtained.

4.20 - Application of Power Curve Equations to Cross Sections:- The use of power curve equations to develop curves for cross sections of a body entails the same procedures as have been outlined previously. However, to familiarize the reader with the shape of the curves which can be obtained, and to further demonstrate the use of the two term power curve equation chart of Fig. 17, a few illustrative examples will be given. Two term equations as derived from Fig. 17 will be exclusively used, although it is entirely possible, in some cases, to develop and use a single term equation, or a two or more term equation

developed by the trial and error method of Section 2.60.

Let us assume that, in Fig. 30, it is required to develop a portion of a cross section so that the line starts at point O and passes through point P, at which (x) = 20, (y) = 20, and the slope is zero.

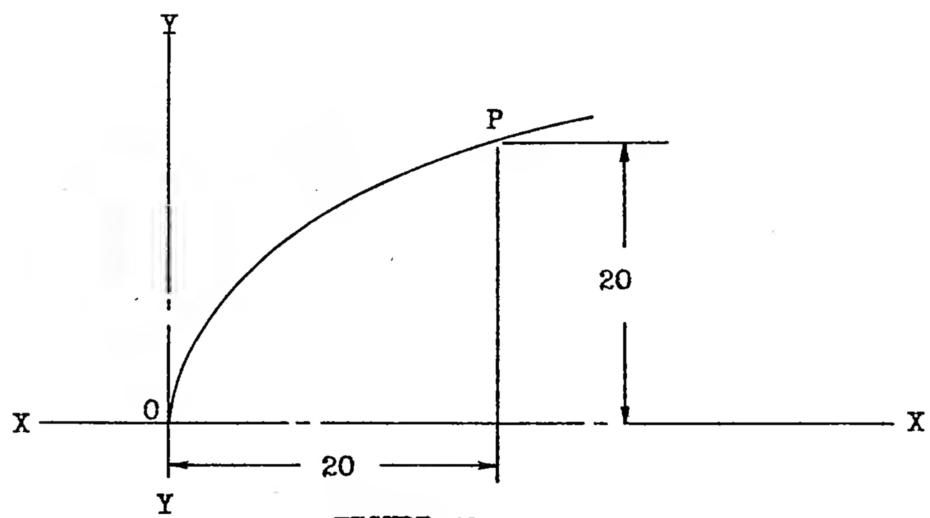


FIGURE 30

Now, obviously, a single term equation could not be developed in this case, since it is impossible for a single term equation to have a zero slope at any point. Hence, the chart of Fig. 17 can readily be used, as it has been developed as a function of the maximum ordinate, at which point the slope is zero.

Suppose, for a first try, that the first and second term exponents are assumed to be 0.5 and 0.8, respectively. The equation so far is then:

$$y = k_1(x)^{0.5} - k_2(x)^{0.8} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (100)$$

From Fig. 17, the maximum ordinate would be at 20.7% of the chord. For the purpose of determining the fundamental curve shape, and solving for (k_1) and (k_2), two points must be evaluated. If the chord is assumed to be 100 units, at which (y) = 0, then Eq. (100) can be evaluated for the maximum ordinate

position and the end of the chord. The maximum ordinate of 20 then occurs at $(x) = 20.7$. Hence:

$$y = 20 = k_1(20.7)^{0.5} - k_2(20.7)^{0.8} \quad \text{.....................(101)}$$

and:

$$y = 0 = k_1(100)^{0.5} - k_2(100)^{0.8} \quad \text{.....................(102)}$$

Solving Eqs. (101) and (102) simultaneously:

$$k_2 = 2.93224$$
$$= N(0.46720)$$

and:

$$k_1 = 11.67354$$
$$= N(1.06720)$$

Eq. (100) then becomes:

$$y = N(1.06720)(x)^{0.5} - N(0.46720)(x)^{0.8} \quad \text{................(103)}$$

Now, if the first term exponent is assumed to be 0.5 and that of the second term to be 1.0, going through the same procedure results in the equation:

$$y = N(0.90309)(x)^{0.5} - N(0.90309-1)(x)^{1.0} \quad \text{................(104)}$$

with the maximum ordinate occuring at 25% of the chord. For first and second term exponents of 0.5 and 2.0, respectively, the maximum ordinate occurs at 39.6% of the chord, and the equation is:

$$y = N(0.62666)(x)^{0.5} - N(0.62666-3)(x)^{2.0} \quad \text{................(105)}$$

With first and second term exponents of 0.5 and 3.0, respectively, the maximum ordinate is at 49% of the chord, and the equation is:

$$y = N(0.53580)(x)^{0.5} - N(0.53580-5)(x)^{3.0} \quad \text{................(106)}$$

Changing the exponent of the first term to 0.4, and that of the second term to 1.0, the maximum ordinate is at 21.7% of the chord, and the equation becomes:

$$y = N(0.98817)(x)^{0.4} - N(0.78817\text{-}1)(x)^{1.0} \quad \ldots\ldots\ldots\ldots\ldots\ldots (107)$$

Finally, increasing the exponent of the first term to 0.6, and leaving the second equal to 1.0, the maximum ordinate occurs at 28% of the chord, and the equation is then:

$$y = N(0.83174)(x)^{0.6} - N(0.03173)(x)^{1.0} \quad \ldots\ldots\ldots\ldots\ldots\ldots (108)$$

Eqs. (103) through (108) are plotted on Fig. 31. To get a more comprehensive comparison, however, it is necessary to change the chord in such a way that the maximum ordinate position in each case falls on station (x) = 20. Each one of the chords will have to be foreshortened in proportion to its maximum ordinate position, the ordinates remaining the same while the abscissae are changed in the required proportion. Thus each value of (x) in each equation is multiplied by a factor equal to twenty divided by the value of (x) at the maximum ordinate position. For example, in Eq. (103), this factor will be equal to 20/20.7, or 0.9662, and in Eq. (104) the factor is 20/25, or 0.80.

This has been done in Fig. 32, where all curves are plotted together to give a direct comparison. It can clearly be seen that, in the equations with an exponent of 0.5 in the first term, the one with the lowest exponent in the second term gives the more bulgy curve, while the sections nearest the origin become more slender as the power of the second term is increased. It will also be noticed, by comparison of Eqs. (104), (107), and (108), that if the exponent of the first term is decreased, the curve bulges more near the origin even if the exponent of the second term is maintained constant.

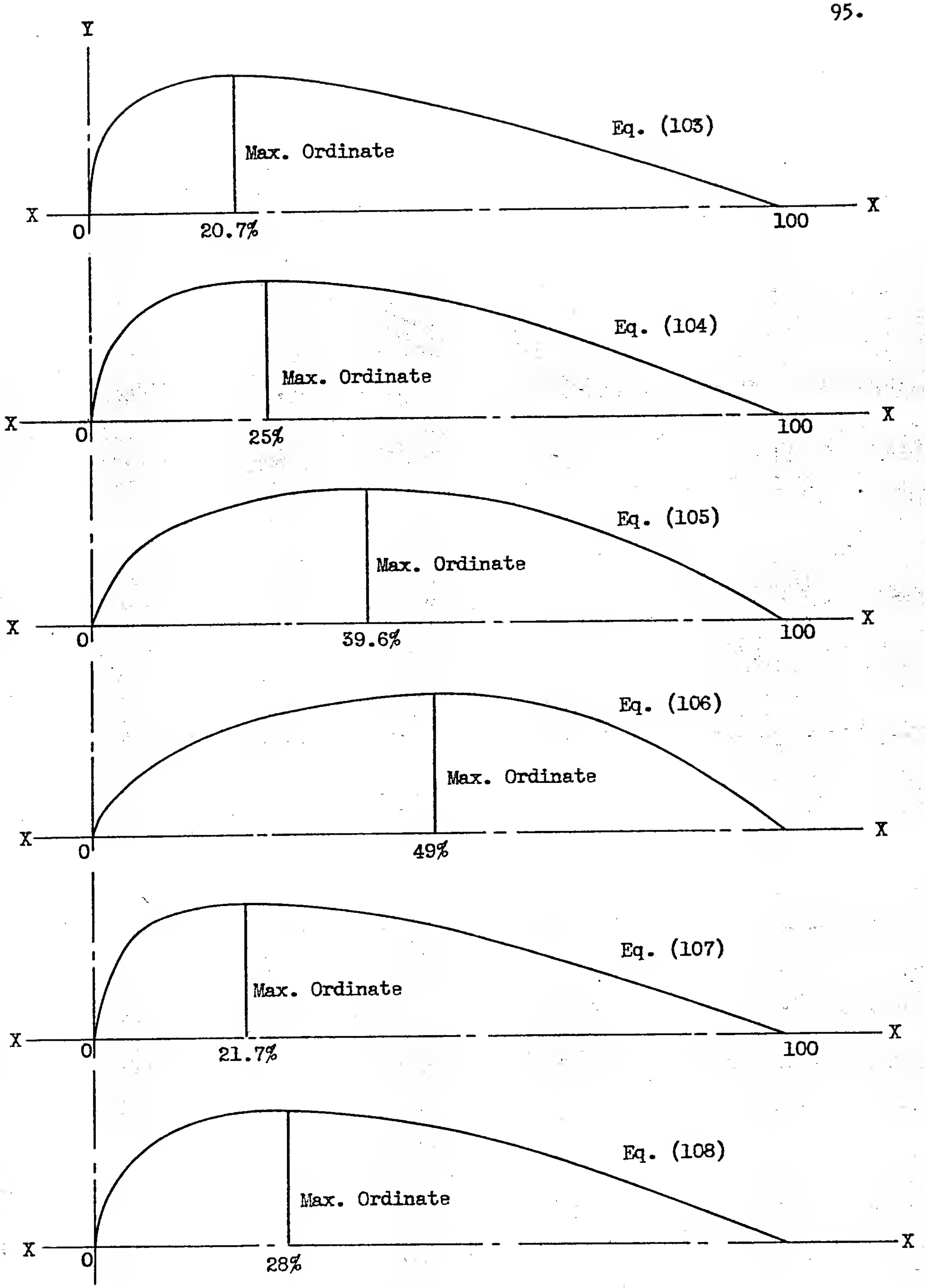

Y
Max. Ordinate
Eq. (103)
X
0
20.7%
100
X
Eq. (104)
Max. Ordinate
X
0
25%
100
X
Eq. (105)
Max. Ordinate
X
0
39.6%
100
X
Eq. (106)
Max. Ordinate
X
0
49%
X
Eq. (107)
Max. Ordinate
X
0
21.7%
100
X
Eq. (108)
Max. Ordinate
X
0
28%
X

FIGURE 31

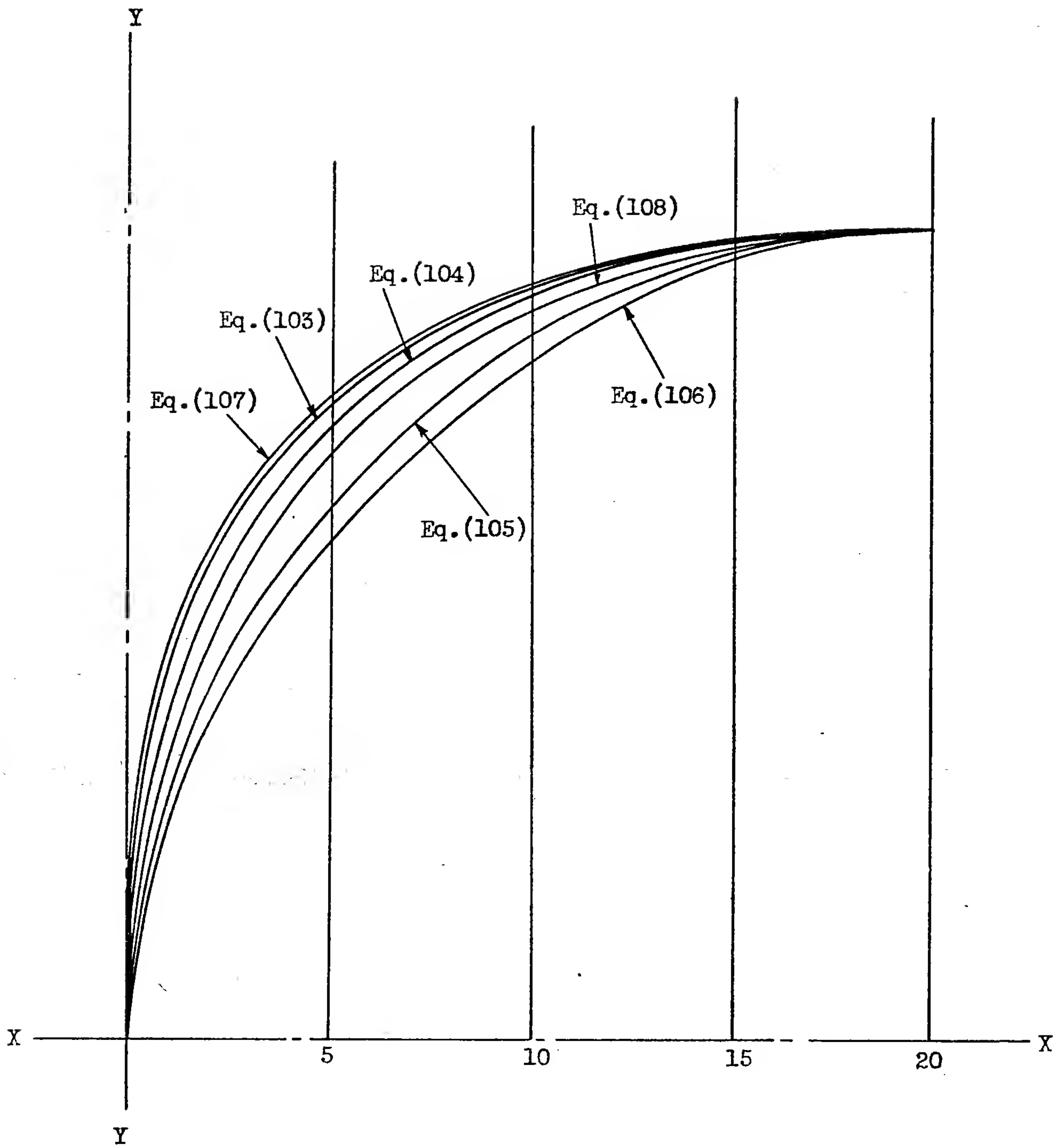


FIGURE 32

The above method outlines the simple manner in which two term power curve equations derived from Fig. 17 can be used for cross section shapes. As mentioned previously, single term equations can also be used, provided a zero slope is not desired, or two or more term equations can be developed by the

trial and error method of Section 2.60. Fig. 32 will serve as a guide, however, to the type of shapes obtainable through the use of two term equations derived from Fig. 17.

As a further illustrative example in the use of Fig. 17, suppose that a curve is to be developed to pass through points P and Q in Fig. 33, the slope at point Q being zero.

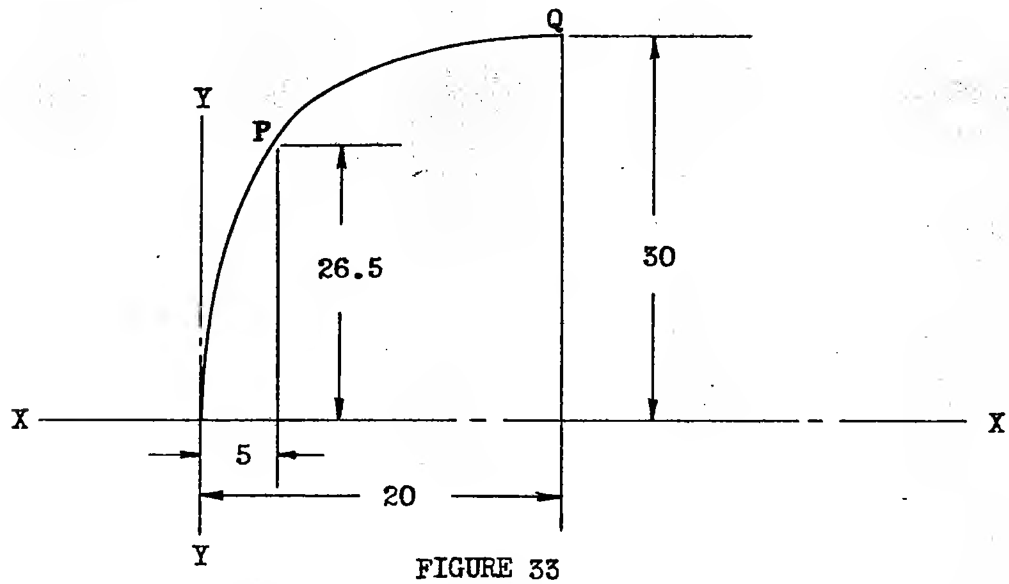


FIGURE 33

For a first assumption, let the first term exponent be 0.3, and the second term exponent be unity. Then, from Fig. 17, the maximum ordinate of a two term equation would be at 17.9% of the chord. Since, in this case, this maximum ordinate position must coincide with $(x) = 20$, the chord is:

$$= (20)(100)/(17.90)$$

$$= 111.69$$

Then, when $(y) = 30$ at $(x) = 20$, and $(y) = 0$ at $(x) = 111.69$, the equations from which (k_1) and (k_2) can be found are:

$$k_1(20)^{0.3} - k_2(20) = 30$$

and:

$$k_1(111.69)^{0.3} - k_2(111.69) = 0$$

Solving these two equations simultaneously, we get:

$$k_2 = 0.64285$$
$$= N(0.80811-1)$$

and:

$$k_1 = 17.4467$$
$$= N(1.24163)$$

The equation of the curve is then:

$$y = N(1.24163)(x)^{0.3} - N(0.80811-1)(x) \qquad \text{..................(109)}$$

Evaluating Eq. (109) for (x) = 5, we find that (y) = 25.06 instead of the required value of 26.5. If, for a second assumption, the first term exponent is left at 0.3, and the second term exponent is reduced to 0.6, the equation becomes:

$$y = N(1.38798)(x)^{0.3} - N(0.69644)(x)^{0.6} \qquad \text{..................(110)}$$

For (x) = 5, the value of (y) in Eq. (110) is found to be 26.547, which will be considered a close enough check.

Here again, the trial and error method of Section 2.60 could have been used, but very often it will be found that considerable time can be saved by the use of Fig. 17, as was done in the illustrative example. The use of either method is optional, however, and one will soon find that practice will dictate the method most often used.

4.30 - Transition Curves for Convergent Lines:- For developing a curved transition for convergent lines, circular arcs are sometimes employed. While

these are easily computed and constructed, they are very often undesirable where it is desired to maintain a smooth fluid flow over the surface, or in highway or railroad curves. This is because the straight lines to be joined obviously have zero curvature, while circular arcs have constant curvature, and therefore the transition from one to the other becomes abrupt.

To avoid this, power curve equations can be developed that start with zero curvature, gradually increase to a maximum value of curvature, and then decrease in the same manner, finally ending up with zero curvature. Multi-term equations can be developed for this type of application by the method given in Section 2.60, but the process is laborious, and, in this case, is not needed. It is easier to develop a single term equation to fit the curve up to its mid-point, and then plot the curve in reverse for the other half of its length. This is particularly desirable for cases where there is reverse flow over, or on, the surface, such as highways, railroad curves, etc.

To develop such a single term power curve equation, the problem consists basically of finding the equation of a curve that has zero curvature at its origin. Let Fig. 34 represent the problem, with the dotted curve being the line for which the equation is desired, and MN and NO the lines to be joined.

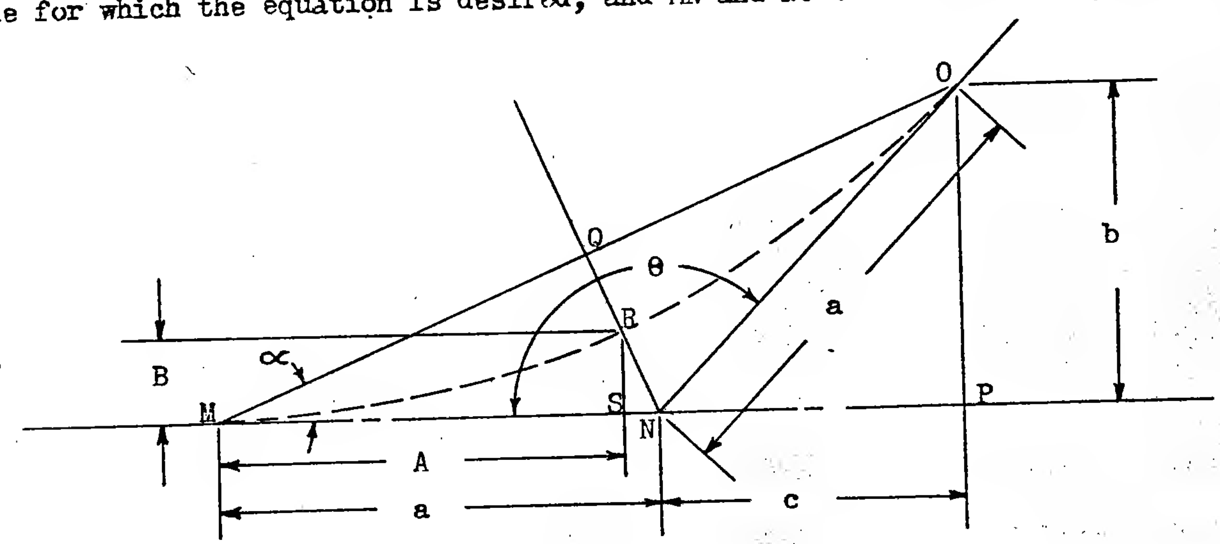


FIGURE 34

On the bisector of the angle θ, point R represents the mid-point of the curve. Then the equation for the first half of the curve starts at point M and ends at point R. The slope at point R, where the required curve is parallel to the line MO, is fixed, and is:

$$t = (b)/(a + c) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (111)$$

However, the lengths A and B are not known, and therefore an infinite number of curves could be drawn between points M and R. To fix one of these unknowns, let us first consider the expression for curvature as given in Eq. (93):

$$K = (d^2y/dx^2)/\left[1 + (dy/dx)^2\right]^{1.5} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (93)$$

Since the curvature is required to be zero at the beginning of the curve, it is apparent that it will only be necessary to find an equation whose second derivative starts with zero and then gradually increases. The value of the exponent, as has been demonstrated previously, determines the amount of curvature. However, to begin with, it is obvious that an exponent of unity will not fulfil the requirements, as this exponent defines a straight line. For an exponent of 2, there results:

$$y = k(x)^2$$

$$dy/dx = 2kx$$

$$d^2y/dx^2 = 2k$$

Here the second derivative is a constant, meaning a constant curvature, which is also unsatisfactory. Furthermore, any exponent between 1 and 2 will have a negative second derivative, indicating a curvature that decreases with (x) rather than increasing with it. Hence, the exponent must be larger than 2, and then the curve will start with zero curvature and increase in curvature as

(x) increases. The magnitude of the exponent depends upon the distance (a), in Fig. 34, and the angle of convergency, and can best be found by a few trial computations.

With the value of the exponent, (n), assumed, the values of A and B can be easily found. From Eq. (54) in Section 2.30, for the case of a curve passing through a point with abscissa (A), ordinate (B), and slope (t) at the point, we have:

$$n = At/B \quad \text{.............................} (54)$$

or:

$$B = At/n \quad \text{.............................} (112)$$

In Fig. 34, the triangles RSN and MPO are similar, and hence the slope (t) at point R is also given by:

$$t = (a - A)/B$$

or:

$$B = (a - A)/t \quad \text{.............................} (113)$$

Equating Eqs. (112) and (113):

$$At/n = (a - A)/t$$

or:

$$A = (a)(n)/(t^2 + n) \quad \text{.............................} (114)$$

Hence the value of A can be found from Eq. (114), and the value of B from Eq. (112). Then the value of (k) for the required equation is given by Eq. (53). When the equation has been developed, the curvature should be checked to insure that it has not reached a maximum before reaching point R, otherwise discontinuities in curvature, or reversals, will occur. Should the maximum curvature not occur at point R, a new value of the exponent has to be assumed.

When plotting the entire curve from points M to O, it must be remembered that, for the first half of the curve, the line MN represents the X axis, with point M as the point of origin. For the second half of the curve, the line NO represents the X axis, with point O as the origin.

For example, consider Fig. 35, where it is desired to develop the equation of a curve connecting points M and O, with zero curvature at these points with respect to the lines MN and NO. Other specified dimensions are:

$$a = 200$$

$$b = 132.288$$

$$c = 150$$

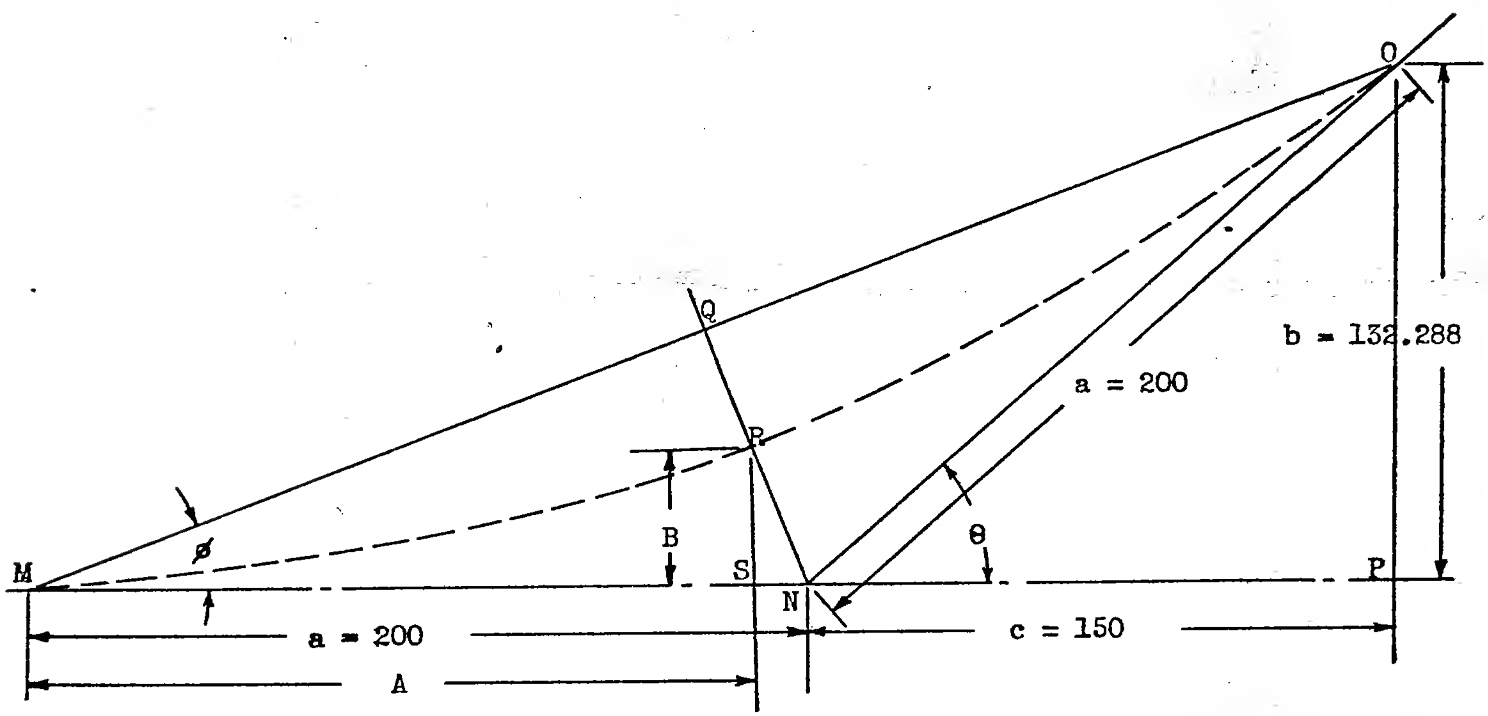


FIGURE 35

From Eq. (111):

$$t = b/(a + c)$$

$$= 132.288/350$$

$$= 0.37796$$

Assuming a value of $(n) = 3.0$, the value of (A) is given by Eq. (114):

$$A = (a)(n)/(t^2 + n)$$

$$= (200)(3)/\left[(0.37796)^2 + 3\right]$$

$$= 190.91$$

From Eq. (112):

$$B = At/n$$

$$= (190.91)(0.37796)/(3)$$

$$= 24.052$$

From Eq. (53):

$$k = B/(A)^n$$

$$= 24.052/(190.91)^3$$

$$= N\left[(\log 24.052) - 3(\log 190.91)\right]$$

$$= N(0.53867{-}6)$$

The equation of the curve then becomes:

$$y = k(x)^n$$

$$= N(0.53867{-}6)(x)^3 \quad \text{.......................}(115)$$

The ordinates, derivatives, and curvatures of Eq. (115) are tabulated in Table XIV.

It can be seen that this curve will be satisfactory in that the curvature continues to increase with (x), and is largest at $(x) = 190.91$, corresponding to point R. The minimum radius of curvature, which occurs at point R, is:

TABLE XIV

x	y	dy/dx	d^2y/dx^2	K
50.00	0.432	0.02593	0.0010571	0.001036
90.00	2.520	0.08401	0.0018668	0.001849
130.00	7.595	0.17527	0.0026965	0.002578
160.00	14.160	0.26550	0.0033187	0.002995
180.00	20.161	0.33602	0.0037336	0.003181
190.00	23.712	0.37439	0.0039410	0.003236
190.91	24.053	0.37798	0.0041484	0.003394

$$R = 1/K$$
$$= 1/0.003394$$
$$= 294.637$$

If an exponent of 2.5 had been assumed to fit the case of Fig. 35, (A) becomes 189.19, (B) becomes 28.603, and the equation is:

$$y = N(0.76418-5)(x)^{2.5} \quad \text{..........................(116)}$$

The ordinates, derivatives, and curvatures of Eq. (116) are given in Table XV below:

TABLE XV

x	y	dy/dx	d^2y/dx^2	K
50.00	1.027	0.05135	0.001541	0.001536
90.00	4.465	0.12401	0.002067	0.002020
130.00	11.177	0.21530	0.002484	0.002321
160.00	18.814	0.29397	0.002756	0.002434
180.00	25.256	0.35077	0.002923	0.002456
189.19	28.603	0.37797	0.002997	0.002453

In this case, it will be noticed that the maximum curvature does not quite occur at point R, $(x = 189.19)$, but is instead at $(x) = 180$. The minimum radius of the curve is:

$$R = 1/0.002456$$
$$= 407.6$$

For an assumed exponent of 4.0, there results:

$$A = 193.10$$
$$B = 18.247$$
$$y = N(0.11802\text{-}8)(x)^4 \quad \text{.........................(117)}$$

Table XVI shows the ordinates, derivatives, and curvatures of Eq. (117):

TA BLE XVI

x	y	dy/dx	d^2y/dx^2	K
50.00	0.082	0.00656	0.000394	0.000394
90.00	0.861	0.03826	0.001275	0.001272
130.00	3.748	0.11532	0.002661	0.002609
160.00	8.600	0.21500	0.004031	0.003767
180.00	13.775	0.30612	0.005102	0.004461
190.00	17.101	0.36003	0.005685	0.004735
193.10	18.247	0.37796	0.005872	0.004806

The maximum curvature now again occurs at point R, as is desired, and the radius of curvature at this point, which is the minimum radius along the curve, is:

$$R = 1/0.004806$$
$$= 208.07$$

The curves defined by Eqs. (115), (116), and (117) are plotted on Fig. (36), and the values of curvature for the three equations are shown on Fig. (37):

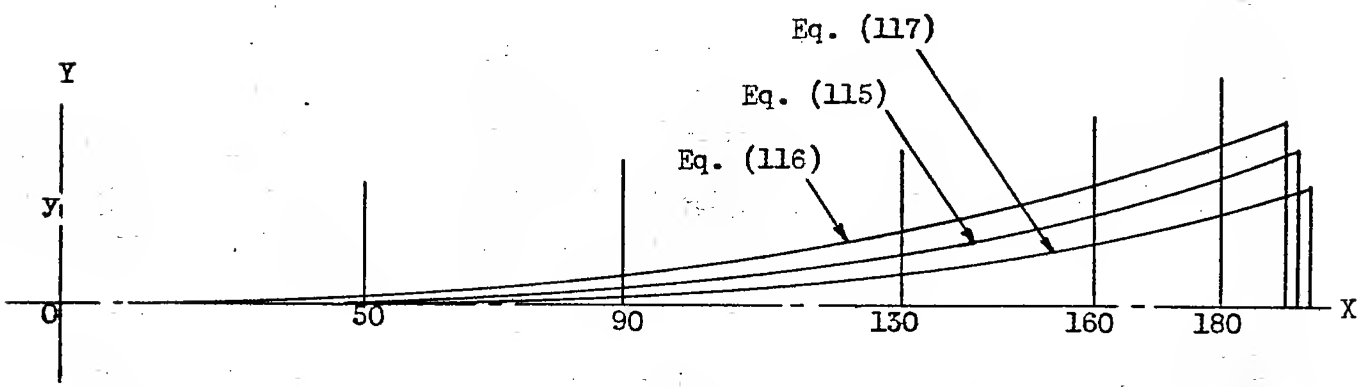


FIGURE 36

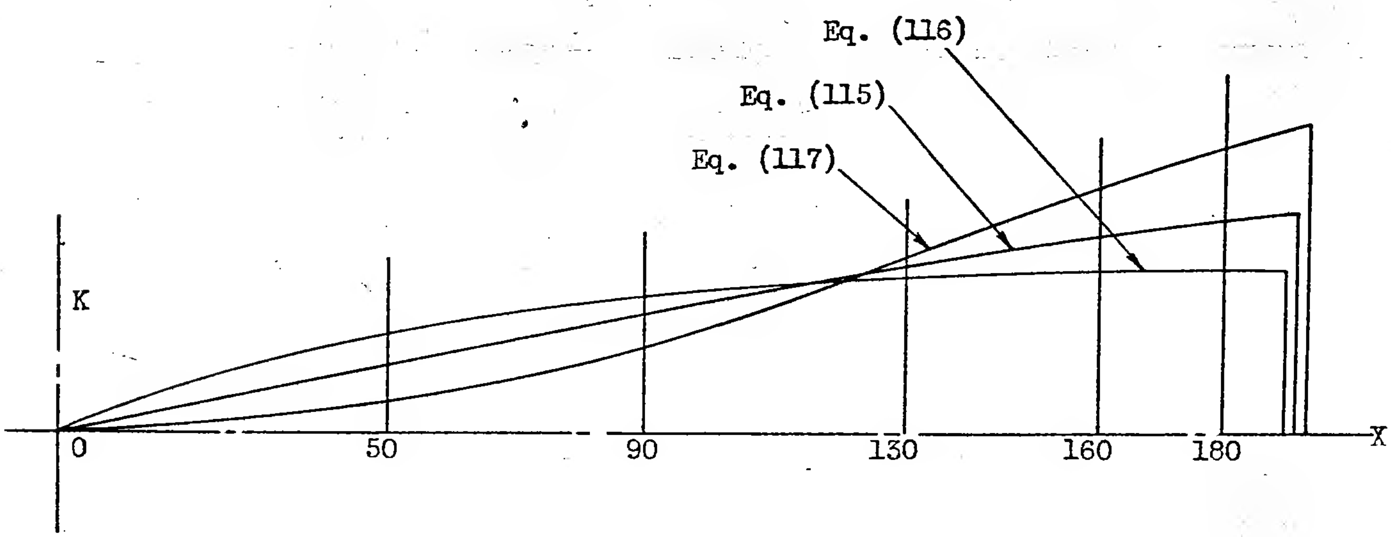


FIGURE 37

By inspecting the curves of Fig. 37, it becomes at once apparent that the curve of Eq. (115), for which the exponent is 3.0, is by far the best, since

it is very close to a straight line, meaning a uniform rate of change of curvature. The equation with an exponent of 4.0, (Eq. 117), is preferred to that with an exponent of 2.5, (Eq. 116), since the increase in curvature is much less at the beginning, and also because the maximum curvature of Eq. (116) actually occurs a short distance before the mid-point of the curve. However, the minimum radius with an exponent of 4.0 is smaller than with an exponent of 2.5, and this may be a point for consideration in some cases.

The above discussion has been based on the procedure of developing the curve so that it is exactly symmetrical about its mid-point, and using different reference lines for each segment of the curve. As mentioned previously, such a development can be used where the flow over, or on, the surface may be in either direction, as in highways, etc. If the flow is in only one direction, however, as is usually the case when dealing with the flow of fluids, the symmetry of curvature about the mid-point may be of lesser importance, and the curve may be developed in another manner which is sometimes easier.

In this method, (a) and (c) are made equal, and drawing a perpendicular through point N will fix (A) equal to (a), as shown in Fig. 38.

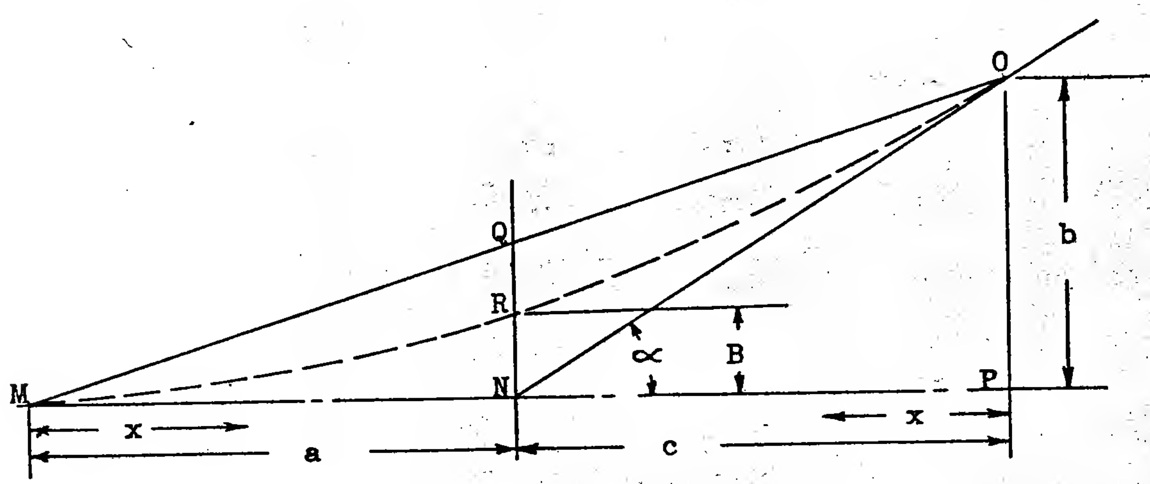


FIGURE 38

The slope at point R is again:

$$t = b/(a + c) \quad \text{..............................(111)}$$

With the value of (A) known, the only unknown is (B), which can be found from Eq. (112), and (k) is found in the usual manner from Eq. (53). After the equation has been developed, and the ordinates computed, the curve up to point R is plotted with line MP as the reference line and point M as the origin.

For the second part of the curve between points R and O, which is plotted with the same reference line but with point P as the origin, the equation developed for the curve MR can also be used providing corrections are made for adding in the ordinates of the line NO. These corrections are needed since the ordinates which would be obtained from the same equation as for line MR would be measured from line NO, and it is desired, for the sake of ease of plotting, to use line NP as the reference line. The value of the ordinates of line NO is a function of (x), as measured from point P, and is found as follows:

$$\tan(\alpha) = b/c$$

or:

$$b = (c)\tan(\alpha)$$

At any point (x):

$$\begin{aligned} y &= (c - x)\tan(\alpha) \\ &= (c - x)(b/c) \\ &= b - (b/c)(x) \quad \text{..............................(118)} \end{aligned}$$

The required equation for the curve RO is then:

$$y = k(x)^n + \left[b - (b/c)(x)\right] \quad \text{..........................(119)}$$

The exponent (n) is assumed again, as in the first case where the

actual mid-point of the curve is used. After the equation has been developed, the curvature should be checked to insure that the maximum value occurs at point R. If such should not be the case, a new value for the exponent should be assumed, and the development repeated.

As an illustrative example of this procedure, consider Fig. 38 and assume the following to be given:

$$a = c = A = 100$$

$$b = 44$$

The slope at point R is then, from Eq. (111):

$$\begin{aligned} t &= b/(a + c) \\ &= 44/(100 + 100) \\ &= 0.22 \end{aligned}$$

Assuming an exponent of 3.0, and using Eq. (112):

$$\begin{aligned} B &= At/n \\ &= (100)(0.22)/(3) \\ &= 7.333 \end{aligned}$$

and, from Eq. (53):

$$\begin{aligned} k &= B/(A)^n \\ &= 7.333/(100)^3 \\ &= N(0.86530\text{-}6) \end{aligned}$$

The equation for the curve MR is now:

$$y = N(0.86530\text{-}6)(x)^3 \quad \text{.........................} (120)$$

The first and second derivatives of Eq. (120) are:

$$dy/dx = N(0.34242\text{-}5)(x)^2 \quad \text{........................} (121)$$

and:

$$d^2y/dx^2 = N(0.64345\text{-}5)(x) \quad \text{.........................(122)}$$

The ordinates, derivatives, and the curvature of Eq. (120) are given in Table XVII below:

TABLE XVII

x	y	dy/dx	d^2y/dx^2	K
0	0.0	0.0	0.0	0.0
30	0.198	0.0198	0.00132	0.00132
60	1.584	0.0792	0.00264	0.00262
80	3.755	0.1408	0.00352	0.00342
90	5.346	0.1782	0.00396	0.00378
100	7.333	0.2200	0.00440	0.00410

For the second portion of the curve, RO, we get, from Eq. (119):

$$y = N(0.86530\text{-}6)(x)^3 + \left[44 - 0.44(x)\right] \quad \text{..................(123)}$$

and the first and second derivatives are:

$$dy/dx = N(0.34242\text{-}5)(x)^2 - 0.44 \quad \text{....................(124)}$$

and:

$$d^2y/dx^2 = N(0.64345\text{-}5)(x) \quad \text{......................(125)}$$

Table XVIII lists the ordinates, derivatives, and curvatures as obtained from Eqs. (123), (124), and (125), for the curve RO.

While the curvature for the curve RO starts with zero, and ends up at point R with a value of 0.00410, the same as for the curve MR, the intermediate values are slightly greater due to the somewhat shorter length of the second

TABLE XVIII

x	y	dy/dx	d^2y/dx^2	K
0	44.000	-0.4400	0.0	0.0
30	30.998	-0.4202	0.00132	0.00103
60	19.184	-0.3608	0.00264	0.00220
80	12.555	-0.2992	0.00352	0.00309
90	9.746	-0.2618	0.00396	0.00359
100	7.333	-0.2200	0.00440	0.00410

part of the curve.

The values of curvature should now be plotted to determine their general shape. After doing so, it may be found that the rate of change of curvature may be improved by the assumption of another value for the exponent. It will be found in all cases that it is well worth the time to plot the curvatures, as sometimes undesirable characteristics will be brought out that are not readily apparent from tabular data.

4.40 - Transition Curves for Parallel Lines:- In the previous section the development of transition curves for convergent lines was demonstrated. For parallel lines, the procedure is the same after a diagonal has been drawn through the parallel lines in such a way that two sets of identical convergent lines are obtained. However, before an equation can be developed, it is first necessary to find a line, or lines, that will divide the system into two identical patterns without an excessive slope. To determine this line, let point M and point T be the points of tangency, as shown in Fig. 39, and let (a) be the length,and (b) the distance between, the two parallel lines.

Then the line PE can be drawn perpendicular to the two lines to be joined, MU and WT, so that MP = PU = (a/2). Point S can then be so located on line

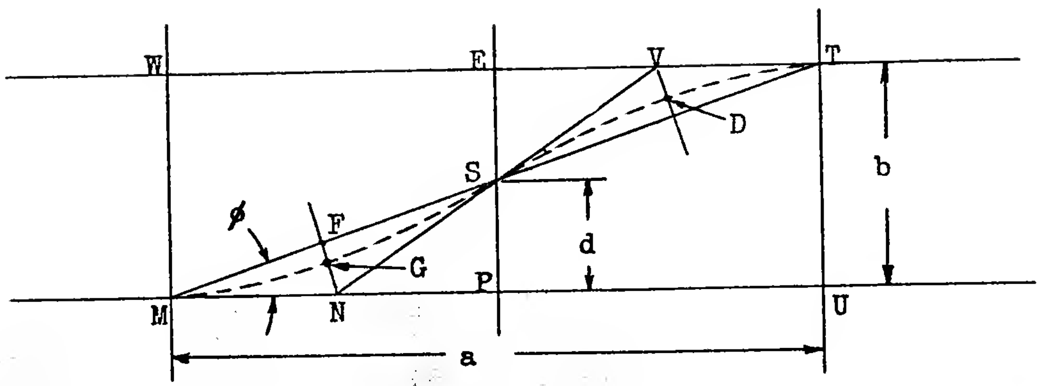


FIGURE 39

PE that SP = SE = (b/2). The problem is now to locate line NSV so that all reference lines, MN, NS, SV, and VT, are equal. To do this, the diagonal line MST is first drawn connecting points M and T, and forming an angle ϕ with lines MU and WT. The length of the line MS is:

$$MS = \sqrt{(MP)^2 + (PS)^2}$$

and the angle ϕ is:

$$\tan(\phi) = SP/MP$$

Considering now the triangle MFN, where FN is a perpendicular to line MS drawn through the mid-point of line MS, it is seen that:

$$\cos(\phi) = MF/MN$$

and therefore the required length MN is:

$$MN = (MF)\cos(\phi) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (126)$$

Lines MN, NS, SV, and VT will now be of equal length, and, if we now

consider the convergent lines MNS and SVT, it is readily seen that each of these is the same case as that of the convergent lines considered in Fig. 35, Section 4.30. Hence, the equation for the curve MG may be found by the first method outlined in Section 4.30, (when using the different reference lines of equal lengths), the slope at point G being equal to (b/a). This curve is used four times, as each quarter will use the same curve. The first point of origin will be at point M, with reference line MN, the second point of origin will be at point S, with NS as the reference line, and so forth.

The problem may also be solved by use of the second method given in Section 4.30, where point G lies on the perpendicular to line MP drawn through the mid-point of line MP, and only one reference line is used. Once the equation for the first section of the curve, MG, has been determined in the form $y = k(x)^n$, the equation for the second quarter becomes:

$$y = k(x)^n + \left[(b/2) - (2b/a)(x)\right] \quad \text{.......................(127)}$$

For the third quarter, the equation is:

$$y = -k(x)^n + \left[(b/2) + (2b/a)(x)\right] \quad \text{.......................(128)}$$

and for the fourth quarter:

$$y = b - k(x)^n \quad \text{..................................(129)}$$

All curves are plotted from the reference line MU, and have respective origins at points M, S, S, and T.

4.50 - <u>Transition Curves for Divergent Lines</u>:- Transition curves for divergent lines can also be developed in a similar manner as demonstrated in Section 4.30 for convergent lines, either by using different reference lines of equal length, as shown in Fig. 34, or by the "mid-span" method, with one

reference line, of Fig. 38.

Consider first the method of different reference lines, as shown in Fig. 40, where lines MS and QG are to be joined by a smooth curve tangent at points M and G. It can be seen that the whole problem actually resolves itself

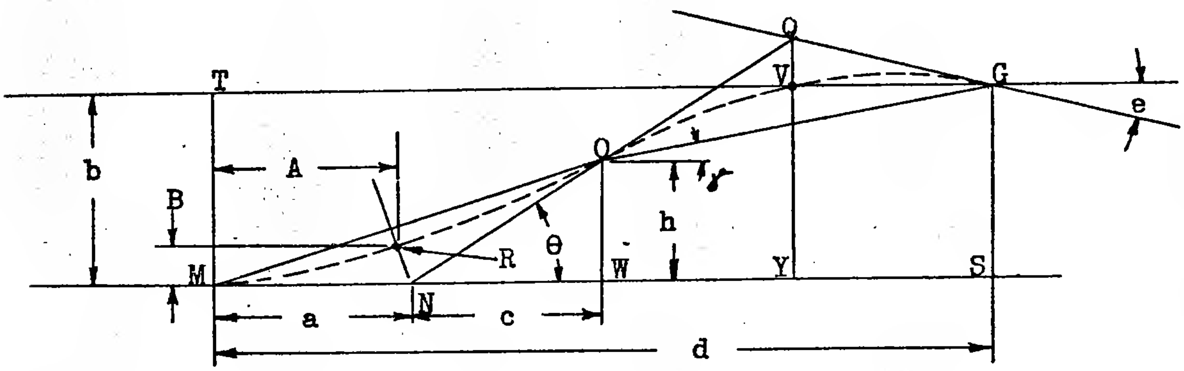


FIGURE 40

into finding a diagonal line so that the distances MN, NO, OQ, and QG are equal, since, once this line has been found, the equation for the curve for each quarter can be found exactly as outlined in Section 4.30. It should be noted, however, that the equations for the curve from points M to O, and from points O to G, **will not** be the same in all cases, as point O will not lie on a straight line through points M and G.

If the distance MN be called (a), then:

$$MN = NO = OQ = QG = (a)$$

The following equalities can be set up by reference to Fig. 40:

$$QV = (QG)\sin(e)$$
$$= (a)\sin(e)$$

$$NY = \sqrt{(NO + OQ)^2 - (QY)^2}$$

$$= \sqrt{(2a)^2 - (b + (a)\sin(e))^2}$$

$$= \sqrt{4a^2 - (b + (a)\sin(e))^2}$$

$$VR = YS = (a)\cos(e)$$

Therefore:

$$d = (a) + (a)\cos(e) + \sqrt{4a^2 - (b + (a)\sin(e))^2} \quad \text{..............}(130)$$

Eq. (130) may be re-arranged as follows:

$$\left[(1 + \cos(e))^2 + \sin^2(e) - 4\right]a^2 + \left[(2b)\sin(e) - 2d(1 + \cos(e))\right]x + b^2 + d^2 = 0 \text{..}(131)$$

This is a quadratic equation which may be solved by means of the standard quadratic formula:

$$a = -E \pm \sqrt{E^2 - 4DF}\ /(2D) \quad \text{........................}(132)$$

where:

$$D = \left[(1 + \cos(e))^2 + \sin^2(e) - 4\right] \quad \text{....................}(132a)$$

$$E = \left[(2b)\sin(e) - 2d(1 + \cos(e))\right] \quad \text{....................}(132b)$$

$$F = (b^2 + d^2) \quad \text{..............................}(132c)$$

Also:

$$\sin(\theta) = QY/NQ$$

$$= (b + (a)\sin(e))/2a \quad \text{.................}(133)$$

$$h = (a)\sin(\theta) \quad \text{...............................}(134)$$

Thus the value of (a) needed to form four equal reference lines, MN, NO, OQ, and QG, can be found from Eq. (132), and the values of (θ) and (h) found from Eqs. (133) and (134), respectively. Then, by drawing the line NR

to bisect the angle MNO, the mid-point of the curve is located at point R. The slope of the curve at point R is:

$$t = h/(a + c)$$

and the values of (A), (B), and (k) can be found, as outlined in Section 4.30, by means of Eqs. (114), (112), and (53), respectively, after assuming the value of the exponent (n). The equation can then be put in the usual $y = k(x)^n$ form.

As an example in the use of this method, assume that the following are given with respect to Fig. 40:

$$d = 850$$

$$b = 200$$

$$e = 10^\circ$$

It is then first necessary to find the required value of (a) as given by Eq. (132). Hence, from Eqs. (132a), (132b), and (132c), we have:

$$\begin{aligned} D &= \left[(1 + \cos(e))^2 + \sin^2(e) - 4\right] \\ &= \left[(1.9848)^2 + (0.1736)^2 - 4\right] \\ &= -0.0304 \end{aligned}$$

$$\begin{aligned} E &= \left[(2b)\sin(e) - (2d)(1 + \cos(e))\right] \\ &= \left[(400)(0.1736) - (1700)(1.9848)\right] \\ &= -3304.72 \end{aligned}$$

$$\begin{aligned} F &= (b^2 + d^2) \\ &= (200)^2 + (850)^2 \\ &= 762{,}500 \end{aligned}$$

Then:

$$a = \frac{-(-3304.72) \pm \sqrt{(-3304.72)^2 - 4(-0.0304)(762500)}}{(2)(-0.0304)}$$

$$= 230.243$$

From Eq. (133):

$$\sin(\theta) = \frac{(200) + (230.243)(0.1736)}{(2)(230.243)}$$

$$= 0.52115$$

From trigonometric tables, corresponding to the above value of $\sin(\theta)$:

$$\cos(\theta) = 0.85347$$

$$\tan(\theta) = 0.61062$$

Also, referring to Fig. 40:

$$NW = (a)\cos(\theta)$$

$$= (230.243)(0.85347)$$

$$= 196.505$$

$$WY = 196.505$$

$$YS = (a)\cos(e)$$

$$= (230.243)(0.98481)$$

$$= 226.746$$

$$h = (a)\sin(\theta)$$

$$= (230.243)(0.52115)$$

$$= 119.991$$

These dimensions are shown on Fig. 41 for reference.

We may now proceed with the development of the equation for the curve

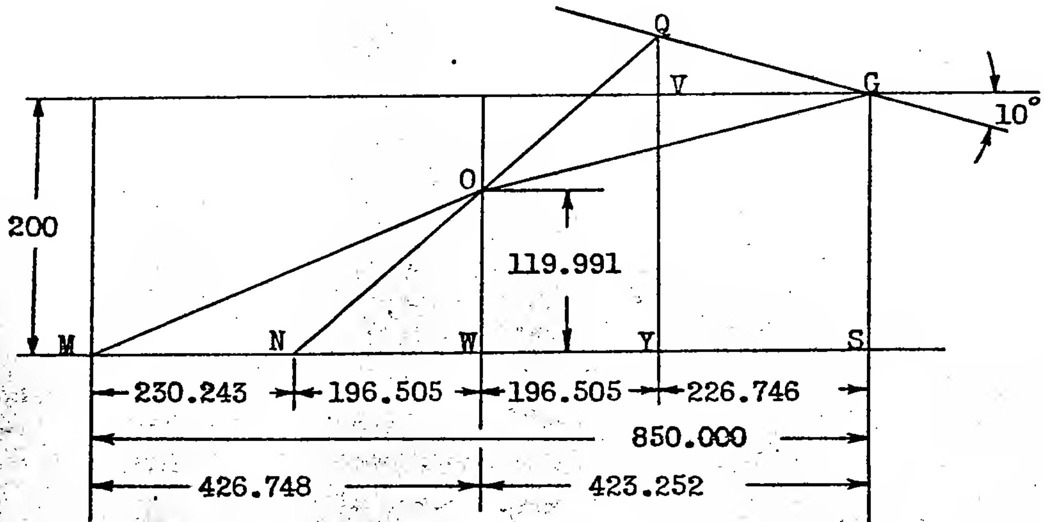


FIGURE 41

between points M and O. The slope at point R, (Fig. 40), is given by Eq. (111), except that, in this case, (b) in Eq. (111) is represented by (h). Hence:

$$t = h/(a + c)$$
$$= (119.991)/(426.748)$$
$$= 0.28118$$

Assuming an exponent of 3.0, the value of (A) is found from Eq. (114):

$$A = (a)(n)/\left[(t)^2 + (n)\right]$$
$$= (230.243)(3)/\left[(0.28118)^2 + 3\right]$$
$$= 224.331$$

From Eq. (112):

$$B = At/n$$
$$= (224.331)(0.28118)/(3)$$
$$= 21.026$$

and, from Eq. (53):

$$k = B/(A)^n$$

$$= N\left[(\log 21.026) - 3(\log 224.331)\right]$$

$$= N(7.32276\text{-}6 - 7.05267)$$

$$= N(0.27009\text{-}6)$$

The equation for the curve between points M and R, and points R and O, is then:

$$y = N(0.27009\text{-}6)(x)^3 \quad \text{.........................(135)}$$

This curve is plotted with the reference line MN, and the point of origin at M. For the curve between points O and G, Fig. 40, it is first necessary to find the slope of the line OG with respect to the new reference line OQ. The slope of the line OQ has already been found:

$$\sin(\theta) = 0.52115$$

$$(\theta) = 31^\circ 24' 34''$$

Also:

$$\tan(\gamma) = (b - h)/(WY + YS)$$

$$= (200.00 - 119.991)/(196.505 + 226.746)$$

$$= 0.18903$$

$$(\gamma) = 10^\circ 53' 47''$$

Hence the angle between lines OG and OQ is:

$$= (\theta) - (\gamma)$$

$$= 20^\circ 30' 57''$$

and the slope of line OG with respect to the reference line OQ is:

$$t = \tan(20^\circ 30' 57'')$$

$$= 0.37420$$

Proceeding as in the case for the curve between points M and O, and employing Eqs. (114), (112), and (53):

$$A = (230.243)(3)/\left[(0.37420)^2 + (3)\right]$$

$$= 219.975$$

$$B = (219.975)(0.37420)/(3)$$

$$= 27.438$$

$$k = N\left[\log(27.438) - 3\log(219.975)\right]$$

$$= N(0.41124\text{–}6)$$

and the equation of the second curve is:

$$y = N(0.41124\text{–}6)(x)^3 \quad \text{..........................(136)}$$

It must be remembered that Eq. (135) is used twice to complete the curve from point M to point O, the first segment being between points M and R, with the origin at point M, and the second segment between points R and O, with the origin at point O. Similarly, Eq. (136) is used twice to plot the curve OG, point O being the origin of the first segment, and point G the origin of the second segment.

Tables XIX and XX show the calculations from Eqs. (135) and (136), respectively. The values are given for only one segment of each curve, as quite obviously the second segments in each case are the same, with respect to their reference lines, as the first segments.

The transition curve for divergent lines can also be plotted from one reference line, instead of the multiple reference line method outlined above. To do this, however, the equations must be developed in a different manner. Consider Fig. 42, for example, and let it be supposed that the transition curve

TABLE XIX

x	y	dy/dx	d^2y/dx^2	K
0.0	0.0	0.0	0.0	0.0
100.00	1.862	0.05587	0.001117	0.00111
180.00	10.861	0.18103	0.002011	0.00192
220.00	19.830	0.27042	0.002451	0.00221
224.33	21.025	0.28118	0.002507	0.00224

TABLE XX

x	y	dy/dx	d^2y/dx^2	K
0.0	0.0	0.0	0.0	0.0
100.00	2.578	0.07733	0.001546	0.00153
180.00	15.033	0.25055	0.002784	0.00254
210.00	23.873	0.34104	0.003248	0.00275
219.975	27.438	0.37420	0.003402	0.00280

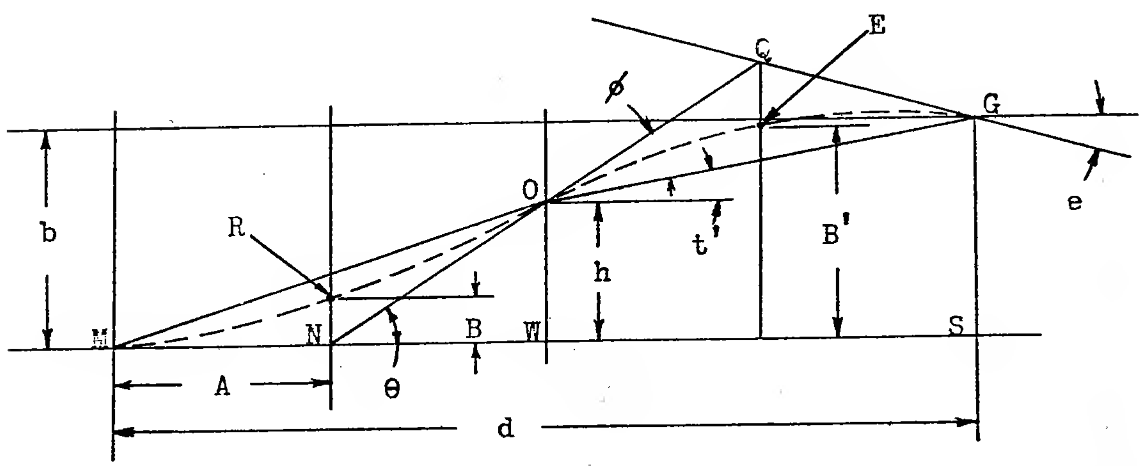


FIGURE 42

indicated by the dotted line, MG, is to be developed so that it can all be plotted from the one reference line MS.

To do this, the distance (d) is divided into four equal parts by the line NQ. The slope of this line is then:

$$\tan(\theta) = \left[\tan(e)(d/4) + b\right]/(d/2) \quad \text{......................(137)}$$

The distance (h), which locates the intersection of the diagonal line NQ with a perpendicular at distance (d/2) from the origin of the curve at point M, is:

$$h = \tan(\theta)(d/4) \quad \text{............................(138)}$$

The slope of the curve MO at its mid-point is then:

$$t = h/(d/2)$$

Hence, having determined these quantities, and assumed an exponent, the equation for the first quarter of the curve, MR, can be found in the usual manner from:

$$A = (d/4)$$

$$B = At/n \quad \text{...............................(112)}$$

$$k = B/(A)^n \quad \text{..............................(53)}$$

and:

$$y = k(x)^n$$

For the second quarter of the curve, RO, it is only necessary to add to the standard curve equation the ordinates of the line NO, since, it will be remembered, the ordinates of this section of the curve with respect to the line NO are the same as the ordinates of the first quarter of the curve with respect to line MN. The origin, in this case, is at point W, the distances (x) being measured to the left from this point. Hence, the curve equation is:

$$y = \left[h - \tan(\theta)x\right] + k(x)^n \quad \text{.........................(139)}$$

For the third portion of the curve, OE, a new equation must be set up, since the slope of the curve at its mid-point, E, is different from that of the curve MO. This slope is equal to the slope of the line OG, and is, with respect to line MS:

$$t' = (b - h)/(d/2) \quad \text{............................(139a)}$$

Since the value of the slope of the curve at its mid-point used in Eq. (112) for finding (B) is given with respect to the basic reference line of the curve, (which is line OQ in this case), the value of (ϕ) represents this slope, and is:

$$\tan(\phi) = \tan(\theta) - t' \quad \text{..........................(139b)}$$

Since (A) is still equal to (d/4), the value of (B') with respect to line OQ is:

$$B' = A \tan(\phi)/n \quad \text{...............................(140)}$$

The ordinates of line OQ, measured from the reference line MS, are given by:

$$y = h + \tan(\theta)x$$

where the origin is at point W and the distances (x) are measured to the right from this origin. Using the value of (B') from Eq. (140) in Eq. (53) to find (k'), the equation for the third portion of the curve, with respect to the reference line MS, becomes:

$$y = h + \tan(\theta)x - k'(x)^n \quad \text{.........................(141)}$$

The origin in this case is at point W. For the curve EG, the equation

will be:

$$y = b + \tan(e)x - k'(x)^n \quad \text{.........................(142)}$$

where the origin is now at point S, and the distances (x) are measured to the left from this point. Here again, in finding the value of (B) in Eq. (112) to use in determining (k') for Eq. (142), care must be taken to refer the slope of line OG to the reference line QG about which the basic equation of the curve is developed. The slope, (t"), is therefore:

$$t'' = t' + \tan(e) \quad \text{.........................(142a)}$$

and therefore:

$$B' = At''/n \quad \text{.........................(142b)}$$

For an example of this procedure, assume the following are given in Fig. 42:

$$d = 850$$

$$b = 200$$

$$e = 10^\circ \ (\tan(e) = 0.17633)$$

$$n = 3.0 \ \text{(assumed)}$$

The slope of the diagonal dividing line, NQ, is, from Eq. (137):

$$\tan(\theta) = \left[(0.17633)(212.5) + (200)\right]/(425)$$

$$= 0.55875$$

and, from Eq. (138):

$$h = (0.55875)(212.5)$$

$$= 118.735$$

The slope of line MO, which is also the slope of the curve MO at its mid-point, is:

$$t = (118.735)/(425)$$
$$= 0.27938$$

Hence, since (A) = (d/4) = 212.5, (B) is, from Eq. (112):

$$B = (212.5)(0.27938)/(3)$$
$$= 19.789$$

From Eq. (53):

$$k = (19.789)/(212.5)^3$$
$$= N(0.31435-6)$$

The equation of curve MR is therefore:

$$y = N(0.31435-6)(x)^3 \quad \text{.........................(143)}$$

For the curve RO, with its origin at point W, it is only necessary to correct the above equation by the amount of the ordinates of line NO. This is done in Eq. (139), and thus the required equation is:

$$y = \left[118.735 - 0.55875(x)\right] + N(0.31435-6)(x)^3 \quad \text{................(144)}$$

In developing the equation for the third portion of the curve, OE, it is first necessary to determine the slope of the line OG, (which is the slope of the curve at its mid-point), with respect to line OQ. This is because the curve equation is actually first developed with respect to line OQ in the standard manner, and then corrections applied to take into account the ordinates of this line. Hence, the slope of line OG is, with respect to line MS:

$$t' = (b - h)/(d/2)$$
$$= (200 - 118.735)/425$$
$$= 0.19121$$

From Eq. (139b), the slope can be corrected to be with reference to line OQ:

$$\tan(\phi) = 0.55875 - 0.19121$$

$$= 0.36754$$

and therefore:

$$B' = (212.5)(0.36754)/(3)$$

$$= 26.034$$

and:

$$k' = (26.034)/(212.5)^3$$

$$= N(0.43346\text{-}6)$$

The equation for the third portion of the curve is then, from Eq. (141):

$$y = \left[118.735 + 0.55875(x)\right] - N(0.43346\text{-}6)(x)^3 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(145)$$

For the fourth, and last, portion of the curve from point E to point G, the same procedure is followed as for the third quarter of the curve. The slope of line OG with respect to line MS is 0.19121, as already found, and therefore, referring it to line QG so that the value of (B) may be found, the slope becomes, from Eq. (142a):

$$t'' = 0.19121 + 0.17633$$

$$= 0.36754$$

From Eq. (142b):

$$B' = (212.5)(0.36754)/(3)$$

$$= 26.034$$

and, using Eq. (53):

$$k' = (26.034)/(212.5)^3$$

$$= N(0.43346-6)$$

Eq. (142) then gives the equation for the fourth portion of the curve, defined by the line EG:

$$y = 200 + 0.17633(x) - N(0.43346-6)(x)^3 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(146)$$

The equations for plotting the four portions of the curve from the one reference line, MS, are then given by Eqs. (143), (144), (145), and (146). The ordinates, derivatives, and curvatures for the four portions are tabulated in Table XXI.

From inspection of Table XXI, it will be seen that the ordinates and curvatures match at all the junction points, indicating a continuous curve. It will be noted, however, that the curvatures for curves MR and RO, and OE and EG, are not exactly the same along the entire length of each respective curve, due to the fact that, when using this method of developing the equations, points R and E are not quite at the mid-spans of their respective curves. This difference in curvature seldom, if ever, is of any consequence, particularly when considering flow on, or over, the developed line in only one direction. However, as has been noted before, the curvatures can be made symmetrical about their exact mid-points by developing the equations in the manner outlined previously, consisting of four separate reference lines.

4.60 - Application of Power Curves to Projectiles and Supersonic Shapes:- Many bodies designed for high speeds require pointed, straight leading edges in order to reduce or delay the effects of compressibility. The lines for these bodies can be developed in the same manner as outlined in Section 2.60, except that the exponent of the first term becomes unity. The first term therefore, is

TABLE XXI

Ordinates, Derivatives, and Curvatures for Transition Curve of Fig. 42

x	y	dy/dx	d^2y/dx^2	K
Curve MR				
0.00	0.000	0.00000	0.000000	0.00000
100.00	2.062	0.06187	0.001237	0.00123
180.00	12.027	0.20045	0.002227	0.00210
200.00	16.499	0.24747	0.002470	0.00227
212.50	19.789	0.27938	0.002630	0.00234
Curve RO				
0.00	118.735	−0.55875	0.000000	0.00000
100.00	64.922	−0.49688	0.001237	0.00089
180.00	30.187	−0.35830	0.002227	0.00186
200.00	23.484	−0.31128	0.002270	0.00216
212.50	19.789	−0.27938	0.002630	0.00234
Curve OE				
0.00	118.735	−0.55875	0.000000	0.00000
100.00	171.897	−0.47736	0.001630	0.00112
180.00	203.488	−0.29504	0.002930	0.00259
200.00	208.780	−0.23318	0.003260	0.00301
212.50	211.436	−0.19121	0.003460	0.00328
Curve EG				
0.00	200.000	−0.17633	0.000000	0.00000
100.00	214.920	−0.09494	0.001630	0.00143
180.00	215.917	−0.08738	0.002930	0.00262
200.00	213.563	−0.14924	0.003260	0.00315
212.50	211.436	−0.19121	0.003460	0.00328

a slope.

For instance, considering Fig. 43, assume that the desired leading edge angle is 18 degrees. Then, since the exponent is unity, the value of the constant (k) for the first term is found from Eq. (53), where (B/A) is equal to the tangent of the angle, or:

$$k = \tan(18^\circ)$$
$$= 0.3249$$

and, in the usual log form:

$$y = N(0.51175-1)(x) \quad \text{.............................(147)}$$

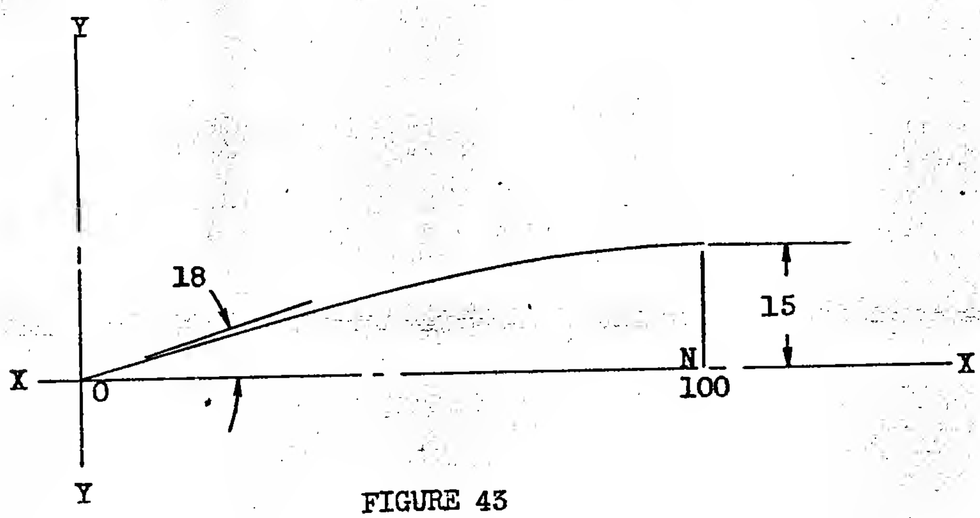


FIGURE 43

Now, if the second point through which the curve must pass is point N, with an ordinate of 15, an abscissa of 100, and a slope of zero, the second term may be found by the method outlined in Section 2.60 for multi-term equations.

Substituting in Eq. (147) to find the value of (y) when (x) is 100:

$$y = 0.3249(100)$$
$$= 32.49$$

Since the required ordinate is 15, the correction to be applied to the

ordinate of the first term equation is, from Eq. (69):

$$C' = E - B'$$
$$= 32.49 - 15.0$$
$$= 17.49$$

The first derivative of Eq. (147) is:

$$dy/dx = 0.3249$$

and, since this is also the slope at (x) = 100 for the first term equation, the slope correction to be applied by the second term of the equation is, from Eq. (71):

$$D' = t - t'$$
$$= 0.3249 - 0$$
$$= 0.3249$$

The required exponent of the second term is found from Eq. (73):

$$n_2 = (A'D')/(C')$$
$$= (100)(0.3249)/(17.49)$$
$$= 1.858$$

and, from Eq. (72):

$$k_2 = (C')/(A')^{n_2}$$
$$= (17.49)/(100)^{1.858}$$
$$= N\left[\log(17.49) - 1.858 \log(100)\right]$$
$$= N(0.52679-3)$$

Since C' and D' are positive, the second term is negative, and then the

equation becomes:

$$y = N(0.51175-1)(x) - N(0.52679-3)(x)^{1.858} \quad \ldots\ldots\ldots\ldots\ldots\ldots (148)$$

More terms can be added, if desired, in order to develop the curve for additional points. This procedure will not be outlined again here, however, as this has already been done in Section 2.60.

CHAPTER V

MENSURATION OF THE LINES, AREAS, AND VOLUMES FORMED BY POWER CURVE EQUATIONS

5.00 - Length of a Line Defined by a Power Curve Equation:- In many cases it may be desired to determine the length of a line which has been plotted from a power curve equation. This can be done mathematically in a few cases, but even then the equations become unduly complicated. Hence, it is far simpler to determine the length graphically, making use of the following equation from the Calculus:

$$S = \int_a^b \left[1 + (dy/dx)^2\right]^{\frac{1}{2}}(dx) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (149)$$

where (S) is the distance from (x) = (a) to (x) = (b), and (dy/dx) the slope of the curve at any distance (x).

Since we usually have the squares of the derivatives from the curvature computations, it is only necessary to add these values to unity, for every station of (x), and plot them against (x). The square root of the area under this curve, which can be computed, or obtained by means of a planimeter, will then give the length of the line to any desired station.

5.10 - Area Under a Power Curve:- The area under any curve developed by a power curve equation may be easily determined mathematically by means of integration. This may be explained by reference to Fig. 44, which shows a curve plotted from the power curve equation:

$$y = k_1(x)^{n_1} - k_2(x)^{n_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (150)$$

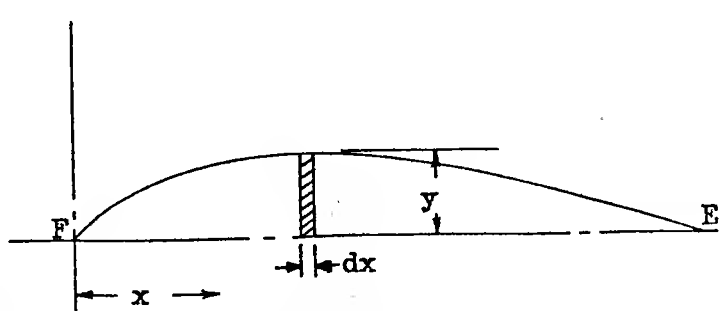


FIGURE 44

If a very small element is considered, with an infinitessimal thickness (dx), then (y) will be the average height, and the area of the element will be (y)(dx). Now, if another element of thickness, (dx), is taken next to the first one, and so on until the entire area is taken, the total area will be the sum of all the little areas. While (dx) is the same for each element, (y) varies, and therefore the total area under the curve will be the sum of all the (y's) multiplied by (dx), or:

$$A = \int y(dx)$$

To determine the area between points F and E, these limits must be specified as follows:

$$A = \int_F^E y(dx) \quad \text{.............................(150a)}$$

Substituting the value of (y) from Eq. (150):

$$A = \int_F^E \left[k_1(x)^{n_1} - k_2(x)^{n_2}\right](dx)$$

$$= \int_F^E k_1(x)^{n_1}(dx) - \int_F^E k_2(x)^{n_2}(dx) \quad \text{................(150b)}$$

Integrating the above, we obtain for the area:

$$A = \left[\frac{k_1(x)^{n_1} + 1}{n_1 + 1}\right]_F^E - \left[\frac{k_2(x)^{n_2} + 1}{n_2 + 1}\right]_F^E \quad \text{.................(151)}$$

To solve the above equation, the value of (x) at station E is first substituted, and the equation solved for (A), the area from (x) = 0 to (x) = E. Then, if the (x) at station F is anything but zero, where (x) is the horizontal distance from the origin of the curve, this value of (x) is substituted and the equation again solved for (A), where this time the area is from (x) = 0 to (x) = F. The final answer, giving the area from (x) = F to (x) = E, is the

difference between these two values of (A). Obviously, if the (x) at station 0 should actually be zero, no second substitution need be made, and the area is that found by substituting only the (x) at station E, since, as noted above, this area is the total area from (x) = 0 to (x) = E.

Eq. (151) applies for a two term equation. For an equation with more terms, the procedure is the same as outlined above, except that additional terms are added.

As an illustrative example, let it be supposed that it is desired to find the area, from (x) = 0 to (x) = 262 inches, under the curve defined by the equation:

$$y = N(0.66636)(x)^{0.4} - N(0.28565\text{-}5)(x)^{2.625}$$

From Eq. (150b), the area is given by:

$$A = \int_0^{262} N(0.66636)(x)^{0.4}(dx) - \int_0^{262} N(0.28565\text{-}5)(x)^{2.625}(dx)$$

Integrating the above equation, the area is given by Eq. (151):

$$A = \left[\frac{N(0.66636)(x)^{1.4}}{1.4}\right]_0^{262} - \left[\frac{N(0.28565\text{-}5)(x)^{3.625}}{3.625}\right]_0^{262}$$

Substituting (x) = 262 inches in the above equation, we can solve for the area from (x) = 0 to (x) = 262 inches as follows:

$$\log(262) = 2.4183$$

$$\log(1.4) = 0.14613$$

$$\log(3.625) = 0.55931$$

$$\log(A) = \left[0.66636 + 1.4(2.4183) - 0.14613\right] - \left[(0.28565\text{-}5) + 3.625(2.4183) - 0.55931\right]$$

$$= 3.90585 - 3.49268$$

$$A = (8051 - 3109) \text{ square inches}$$

$$= 4942 \text{ square inches}$$

For the lower limit of $(x) = 0$, no substitution need be made, and hence the required area is as found above, or 4942 square inches.

5.20 - Center of Gravity of an Area Under a Power Curve:- In the previous section, the area under a curve plotted from a power curve equation was found by summing up the areas of all the little elements between certain limits. If now each little element is multiplied by its distance from a fixed point, the result is the moment of this area about that fixed point. This fixed point, or center of the moment, can be assumed to be located anywhere, but for convenience it is preferably taken at the origin so that the distance from the center of moment will equal some (x).

For each element, then, the moment will be $(x)(y\ dx)$, and, for the entire area, the moment will be:

$$M = \int x(y\ dx) \qquad \text{.................................(152)}$$

But, from Eq. (150a):

$$A = \int y(dx)$$

Therefore, the moment is equal to the area multiplied by the distance (x), or:

$$M = A(x)$$

or:

$$x = M/A \qquad \text{...................................(153)}$$

Eq. (153) then gives the expression for the distance from the center of moment, or origin, to the center of gravity.

Assuming, for example, that the equation of the curve is again expressed as in Eq. (150), and substituting for (y) in Eq. (152), the moment of the area

from (x) = F to (x) = E is:

$$M = \int_F^E \left[k_1(x)^{n_1} - k_2(x)^{n_2}\right](x)(dx)$$

$$= \int_F^E k_1(x)^{n_1 + 1}(dx) - \int_F^E k_2(x)^{n_2 + 1}(dx)$$

Integrating the above, we obtain for the moment:

$$M = \left[\frac{k_1(x)^{n_1 + 2}}{n_1 + 2}\right]_F^E - \left[\frac{k_2(x)^{n_2 + 2}}{n_2 + 2}\right]_F^E \quad \text{.....................(154)}$$

For example, assume that it is desired to find the center of gravity of the area under the same curve as illustrated in the previous section, Fig. 44, and defined by:

$$y = N(0.66636)(x)^{0.4} - N(0.28565\text{-}5)(x)^{2.625}$$

Since the area has already been found, it is now only necessary to find the moment from Eq. (154):

$$M = \left[\frac{N(0.66636)(x)^{2.4}}{2.4}\right]_0^{262} - \left[\frac{N(0.28565\text{-}5)(x)^{4.625}}{4.625}\right]_0^{262}$$

Substituting (x) = 262 in the above, the moment is found as follows:

$$\log(262) = 2.4183$$

$$\log(2.4) = 0.38021$$

$$\log(4.625) = 0.66511$$

$$\log(M) = \left[0.66636 + 2.4(2.4183) - 0.38021\right] - \left[(0.28565\text{-}5) + 4.625(2.4183) - 0.66511\right]$$

$$= 6.09007 - 5.80518$$

$$M = (1{,}230{,}514 - 638{,}528)$$
$$= 591{,}986 \text{ in}^3$$

Since the area under the curve, from $(x) = 0$ to $(x) = 262$, was found in the previous section to be 4942 square inches, the distance of the center of gravity from $(x) = 0$ is found from Eq. (153), or:

$$x = 591986/4942$$
$$= 119.79 \text{ inches}$$

The above example illustrates the procedure for a two term equation. However, the procedure to be followed for three or more term equations is exactly the same as outlined above, except for the addition of more terms in the same manner as the second term was handled above.

5.30 - Surface Area of a Body Developed from a Power Curve:- The surface, or wetted, area of a body is sometimes required in the computation of skin friction drag, or in the computation of the weight of the body. In the case of a body of revolution, it can be determined by simple integration in the following manner:

Consider the body shown in Fig. 45, which represents a body of revolution of which the surface area is to be found.

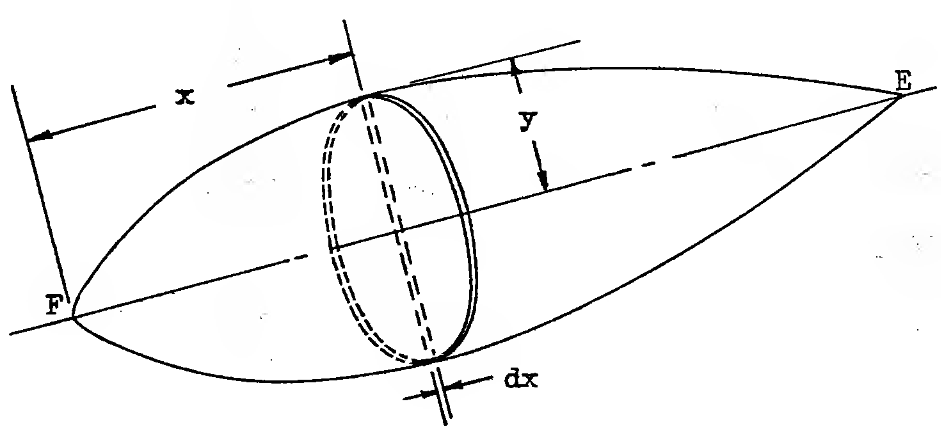


FIGURE 45

For each small element of surface area, with a width (dx), the area is $2\pi(r)(dx)$, where (r) is the radius. However, (r) is equal to (y), the ordinate of the curve at distance (x) from the origin. Hence, summing up all these elemental areas, the total surface area is:

$$W = \int_F^E 2\pi(y)(dx)$$

or, since 2π is a constant:

$$W = 2\pi \int_F^E (y)(dx) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (155)$$

Comparing Eq. (155) with Eq. (150a) in Section 5.10, which gives the area under a power curve, it can be seen that the only difference is the multiplier, 2π. The surface area of a body of revolution developed from a power curve equation is therefore equal to the area under the power curve, as found by Eq. (150a) or (151), multiplied by the constant 2π. Since the procedure for finding the area under the curve has already been outlined and demonstrated in Section 5.10, no additional illustrative examples need be given.

For a body with an elliptical cross-section, where the contour lines in plan and elevation views are developed from two different power curve equations, the exact calculation of the surface area becomes quite complicated. However, a reasonably close approximation can be made, if the difference between the major and minor axes is not too great, in the following manner.

If the plan and elevation view contour lines are developed from two equations, designated (y_1) and (y_2), the surface area is approximately:

$$W = 2\pi \int_F^E \left[(y_1 + y_2)/2\right](dx)$$

$$= \pi\int_F^E y_1(dx) + \pi\int_F^E y_2(dx) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(156)$$

Eq. (156) can then be put in similar form to Eq. (151), the number of terms depending on the number of terms in the equations (y_1) and (y_2). It can then be solved as outlined and demonstrated in Section 5.10, remembering, of course, to multiply by the constant π.

5.40 - Volume of a Body of Revolution Developed from a Power Curve:- The body of revolution shown in Fig. 46 was formed by rotating the line developed from a power curve equation 360 degrees about the X axis. Let it now be assumed that

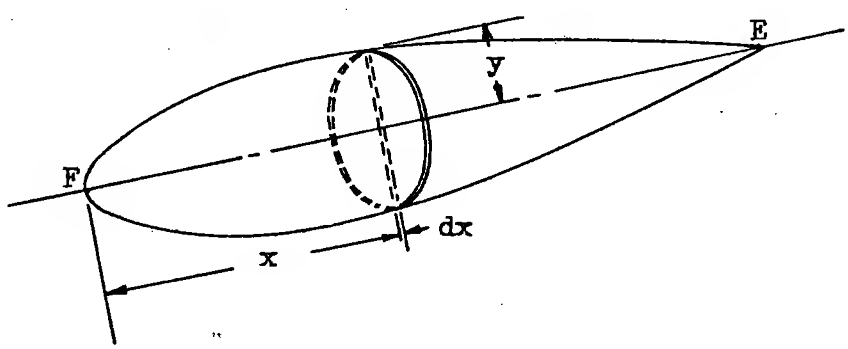


FIGURE 46

the internal volume of the body so formed is to be found.

If a circular disc of infinitessimal thickness, (dx), is considered, with a radius equal to (y), the ordinate of the curve at a distance (x) from the origin, then the volume of this disc is:

$$V = \pi y^2(dx)$$

Now, summing up all such discs along the entire length of the body, the total volume becomes:

$$V = \pi\int_F^E y^2(dx) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(157)$$

If the equation of the power curve equation is of the following form, for example,

$$y = k_1(x)^{n_1} - k_2(x)^{n_2}$$

then, substituting for (y) in Eq. (157):

$$V = \pi\int_F^E \left[k_1(x)^{n_1} - k_2(x)^{n_2}\right]^2 (dx) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(158)$$

The expression in brackets can be squared, as follows:

$$y^2 = (k_1)^2(x)^{2n_1} - 2k_1k_2(x)^{n_1 + n_2} + (k_2)^2(x)^{2n_2} \quad\ldots\ldots\ldots\ldots(158a)$$

and, substituting back in Eq. (158):

$$V = \pi\int_F^E (k_1)^2(x)^{2n_1}(dx) - \pi\int_F^E 2k_1k_2(x)^{n_1 - n_2}(dx) + \pi\int_F^E (k_2)^2(x)^{2n_2}(dx)\ldots(159)$$

Integrating Eq. (159), we get:

$$V = \left[\frac{\pi(k_1)^2(x)^{2n_1+1}}{2n_1+1}\right]_F^E - \left[\frac{2\pi k_1k_2(x)^{n_1+n_2+1}}{n_1+n_2+1}\right]_F^E + \left[\frac{\pi(k_2)^2(x)^{2n_2+1}}{2n_2+1}\right]_F^E \ldots(160)$$

Eq. (160) then gives the expression for the volume of a body of revolution formed by rotating the curve of a two term power equation 360 degrees about the X axis. For basic curve equations with more or less terms than two, Eq. (160) will be of the same form except for the addition or subtraction of more terms, as determined by substitution in the fundamental equation for the volume, Eq. (157).

As an illustrative example, consider the previously demonstrated equation:

$$y = N(0.66636)(x)^{0.4} - N(0.28565-5)(x)^{2.625}$$

Squaring this equation for substitution in Eq. (157), we get, from Eq. (158a):

$$y^2 = N(1.33272)(x)^{0.8} - N(0.25304-4)(x)^{3.025} + N(0.57130-10)(x)^{5.250}$$

If the volume is desired between the same limits as before, (x) = 0 to (x) = 262, then, from Eq. (159):

$$V = \pi\int_0^{262} N(1.33272)(x)^{0.8}(dx) - \pi\int_0^{262} N(0.25304-4)(x)^{3.025}(dx)$$

$$+ \pi\int_0^{262} N(0.57130-10)(x)^{5.25}(dx)$$

By integrating the above, the volume becomes as given in Eq. (160):

$$V = \left[\frac{(\pi)N(1.33272)(x)^{1.8}}{1.8}\right]_0^{262} - \left[\frac{(\pi)N(0.25304-4)(x)^{4.025}}{4.025}\right]_0^{262}$$

$$+ \left[\frac{(\pi)N(0.57130-10)(x)^{6.25}}{6.25}\right]_0^{262}$$

$$\log(V/\pi) = 5.43039 - 9.38193 - 4.88970$$

$$(V/\pi) = 269394 - 240950 - 77587$$

$$= 106031$$

$$V = 333{,}106 \text{ cubic inches}$$

$$= 192.77 \text{ cubic feet}$$

5.50 - Volume of a Body of Elliptical Cross Section Developed from Two Power Curve Equations: - In Section 5.40 the method was demonstrated whereby the volume of a body of revolution could be found where the contour lines were defined by a power curve equation. Somewhat the same procedure is followed in the case of a body of elliptical cross section, where the contour lines in both plan and elevation views are developed from power curve equations. Here, however, instead of squaring the one equation for (y), we multiply together the two equations in (y), and then solve as in Section 5.40.

For example, assume that the two equations defining the contour lines in plan and elevation views are:

$$y_a = k_{a_1}(x)^{n_{a_1}} - k_{a_2}(x)^{n_{a_2}} \quad \text{.......................(161)}$$

$$y_b = k_{b_1}(x)^{n_{b_1}} - k_{b_2}(x)^{n_{b_2}} \quad \text{.......................(162)}$$

The volume is then given by Eq. (163):

$$V = \pi \int_F^E (y_a)(y_b)dx \quad \text{..........................(163)}$$

Substituting Eqs. (161) and (162) in the above:

$$V = \pi \int_F^E k_{a_1}k_{b_1}(x)^{n_{a_1}+n_{b_1}}(dx) - \pi \int_F^E k_{a_2}k_{b_1}(x)^{n_{b_1}+n_{a_2}}(dx)$$

$$-\pi \int_F^E k_{b_2}k_{a_1}(x)^{n_{a_1}+n_{b_2}}(dx) + \pi \int_F^E k_{b_2}k_{a_2}(x)^{n_{a_2}+n_{b_2}}(dx) \quad \text{........(164)}$$

Eq. (164) then gives the final expression for the volume of the body between the arbitrary points F and E. The integration of Eq. (164) is:

$$V = \left[\frac{\pi k_{a_1} k_{b_1} (x)^{n_{a_1} + n_{b_1} + 1}}{n_{a_1} + n_{b_1} + 1} \right]_F^E - \text{etc., etc.} \quad \ldots\ldots\ldots\ldots\ldots (165)$$

As an illustrative example, assume that it is required to find the volume, from $(x) = 0$ to $(x) = 262$ inches, of the body whose plan and elevation contour lines are given by the equations:

$$y = N(0.66636)(x)^{0.4} - N(0.28565{-}5)(x)^{2.625}$$

$$y = N(0.23829)(x)^{0.6} - N(0.73174{-}4)(x)^{2.05}$$

Before setting up the equation in the form of Eq. (164), it is first necessary to multiply out the constants and exponents, as follows:

$$k_{a_1} k_{b_1} (x)^{n_{a1} + n_{b1}} = N(0.66636 + 0.23829)(x)^{0.4 + 0.6}$$

$$= N(0.90465)(x)$$

$$k_{a_2} k_{b_1} (x)^{n_{b1} + n_{a2}} = N\left[(0.28565{-}5) + (0.23829)\right](x)^{0.6 + 2.625}$$

$$= N(0.52394{-}5)(x)^{3.225}$$

$$k_{b_2} k_{a_1} (x)^{n_{a1} + n_{b2}} = N\left[(0.73174{-}4) + (0.66636)\right](x)^{0.4 + 2.05}$$

$$= N(0.39810{-}3)(x)^{2.45}$$

$$k_{b_2} k_{a_2} (x)^{n_{a2} + n_{b2}} = N\left[(0.73174{-}4) + (0.28565{-}5)\right](x)^{2.625 + 2.05}$$

$$= N(0.01739{-}8)(x)^{4.675}$$

Substituting these terms in an equation similar to Eq. (165), the volume is given by:

$$V = \left[\frac{(\pi)N(0.90465)(x)^2}{2}\right]_0^{262} - \left[\frac{(\pi)N(0.52394\text{-}5)(x)^{4.225}}{4.225}\right]_0^{262}$$

$$- \left[\frac{(\pi)N(0.39810\text{-}3)(x)^{3.45}}{3.45}\right]_0^{262} + \left[\frac{(\pi)N(0.01739\text{-}8)(x)^{5.675}}{5.675}\right]_0^{262}$$

For $(x) = 262$ inches, this becomes:

$$\log(V/\pi) = \left[(0.90465) + 2\log(262) - \log(2)\right] - \left[(0.52394\text{-}5) + 4.225\log(262) - \log(4.225)\right] - \left[(0.39810\text{-}3) + 3.45\log(262) - \log(3.45)\right] + \left[(0.01739\text{-}8) + 5.675\log(262) - \log(5.675)\right]$$

$$= 5.44022 - 5.11543 - 5.20342 + 4.98727$$

$$V = (275{,}563 - 130{,}445 - 159{,}743 + 97{,}112)$$

$$= 259{,}140 \text{ cubic inches}$$

5.60 – Proportional Change of Volume of a Body of Revolution:– In many instances, after a body of revolution has been developed from a power curve equation, and the volume checked, it will be found that the volume is smaller or larger than desired, and a new equation is required.

Instead of going through all the processes of developing a new equation, however, the required equation can be more easily found by employing the procedure shown below.

The volume of a body of revolution is equal to the length, (L), multiplied by the average cross-sectional area, which, being circular, is given by:

$$(A)_{avg} = \pi (r)^2_{avg}$$

where (A) is the cross-sectional area and (r) the radius. Since, in power curve equation form, $(r)_{avg}$ is the $(y)_{avg}$, the volume is:

$$V = (\pi)L(y)^2_{avg} \quad \text{.................................(166)}$$

Now, if the letter (f) represents what may be called the "average fineness ratio", then:

$$f = L/(y)_{avg} \quad \text{.................................(167)}$$

or:

$$L = f(y)_{avg} \quad \text{.................................(168)}$$

Substituting Eq. (168) in Eq. (166):

$$V = (\pi)f(y)^3_{avg}$$

or:

$$(y)_{avg} = (V/(\pi)f\ )^{1/3} \quad \text{.............................(169)}$$

If (V_1) represents the volume of the body as first developed, (V_2) the desired volume, $(y_1)_{avg}$ and $(y_2)_{avg}$ the respective average ordinates, then:

$$(y_2)_{avg}/(y_1)_{avg} = (V_2/\pi f_2)^{1/3}/(V_1/\pi f_1)^{1/3} \quad \text{...............(170)}$$

By keeping the "average fineness ratio" of both bodies the same, (f_1) and (f_2) in Eq. (170) cancel out, and $(y)_{max}$ may be substituted for $(y)_{avg}$. Hence:

$$(y_2)_{max}/(y_1)_{max} = (V_2/V_1)^{1/3} \quad \text{.........................(171)}$$

In a similar manner it can be shown that:

$$L_2/L_1 = (V_2/V_1)^{1/3} \quad \text{..............................(172)}$$

Eqs. (171) and (172) then show how the maximum ordinates and overall lengths of two bodies vary, provided the fineness ratios are the same. The equation of the first body can then be corrected by these ratios in the manner out-

-lined in Section 2.70.

For an example, consider again the body of revolution whose volume was found in Section 5.40 to be 333,106 cubic inches, and whose equation is:

$$y = N(0.66636)(x)^{0.4} - N(0.28565-5)(x)^{2.625}$$

The length of this body is 262 inches, and the maximum ordinate of the power curve is 26 inches. (See Section 2.42). Let it now be required to develop a new power equation for a similar body with the volume increased to 369,600 cubic inches. Then, from Eq. (171):

$$(y_2)_{max}/(y_1)_{max} = (369600/333106)^{1/3}$$
$$= N(0.01505)$$

The new maximum ordinate of the power curve will then be:

$$\log(y_2)_{max} = \log(26) + (0.01505)$$
$$= 1.41497 + 0.01505$$
$$= 1.43002$$
$$(y_2)_{max} = 26.917 \text{ inches}$$

From Eq. (172), the correction for length is also the same, and therefore the new length is:

$$\log(L_2) = \log(262) + 0.01505$$
$$= 2.41830 + 0.01505$$
$$= 2.43335$$
$$(L_2) = 271.24 \text{ inches}$$

In Section 2.70 it was shown that, in order to change a power curve equation for a change in thickness ratio, or maximum ordinate, it is necessary to multiply only the constants of the equation by the ratio of the new to the old

ordinates, as the exponents are not affected by changes in thickness. Therefore, the equation becomes, corrected for the differences in $(y)_{max}$ only:

$$y = N(0.66636 + 0.01505)(x)^{0.4} - N\left[(0.28565-5) + (0.01505)\right](x)^{2.625}$$

$$= N(0.68141)(x)^{0.4} - N(0.30070-5)(x)^{2.625} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (173)$$

To correct the values of (x) for the change in length, the multiplier is the ratio of the old to the new lengths, (or the reciprocal of the ratio used for correcting the constants), raised to the same power as (x). This is shown in Section 2.70. In log form, this correction is:

$$= N\left[(1.0000-1) - (0.01505)\right]$$

$$= N(0.98495-1)$$

Inserting this value in Eq. (173):

$$y = N\left[(0.68141) + (0.98495-1)^{0.4}\right](x)^{0.4} - N\left[(0.3007-5) + (0.98495-1)^{2.625}\right](x)^{2.625}$$

$$= N\left[(0.68141) + (0.39398-0.4) + (0.6 - 0.6)\right](x)^{0.4} - N\left[(0.30070-5) + (2.58549 - 2.625) + (0.375 - 0.375)\right](x)^{2.625}$$

$$= N(0.67539)(x)^{0.4} - N(0.26119-5)(x)^{2.625} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (174)$$

Eq. (174) is then the final equation for the desired body of revolution with a volume of 369,600 cubic inches.

5.70 – <u>Proportional Change of Volume of a Body with Elliptical Cross Section</u>: – The procedure used to correct the contour line equations of a body with an elliptical cross section for a change in volume is substantially the same as given in Section 5.60 for a body of revolution. However, in this case, the $(y)_{max}$ used represents the average of the $(y)_{max}$ values for the plan view and elevation view equations, proportioned in the ratio of the major and minor axes.

The $(y)_{max}$ values are here designated $(y)_{major}$ and $(y)_{minor}$.

Thus, if the ratio of the values of $(y)_{major}$ to $(y)_{minor}$ for the two contour lines is 1.5 to 1.0, then:

$$(y)_{major}/(y)_{minor} = (1.5)/(1.0)$$

and the average $(y)_{max}$ is given by:

$$2(y)_{max} = (y)_{major} + (y)_{minor}$$

$$= 1.5(y)_{minor} + (y)_{minor}$$

$$= 2.5(y)_{minor}$$

Then:

$$(y)_{minor} = 2(y)_{max}/2.5$$

and:

$$(y)_{major} = 2.5(y)_{max}/2$$

If the ratio of $(y)_{major}$ to $(y)_{minor}$ is called (p), then:

$$(y)_{minor} = 2(y)_{max}/(p + 1) \quad \text{.........................(175)}$$

$$(y)_{major} = 2(y)_{max}/(\tfrac{1}{p} + 1) \quad \text{.........................(176)}$$

As an example, consider again the elliptical body whose contour lines are given by the equations in Section 5.50:

$$(y)_{major} = N(0.66636)(x)^{0.4} - N(0.28565\text{-}5)(x)^{2.625}$$

$$(y)_{minor} = N(0.23829)(x)^{0.6} - N(0.73174\text{-}4)(x)^{2.05}$$

The volume of this body, as found in Section 5.50, was 259,140 cubic inches. Its length is 262 inches, and the $(y)_{max}$ of the major curve is 26 inches and that of the minor curve 20.8 inches. Let us assume that the volume is to be changed to 231,000 cubic inches, with the ratio of major to minor axes, and the "average fineness ratio", kept unchanged.

From Eq. (171), the ratio of the average $(y)_{max}$ is:

$$(y_2)_{max}/(y_1)_{max} = (231000/259140)^{1/3}$$
$$= N(0.98336-1)$$

The average $(y)_{max}$ of the original body was:

$$(y_1)_{max} = (26.0 + 20.8)/2$$
$$= 23.4 \text{ inches}$$

and hence the average $(y)_{max}$ of the second body will be:

$$(y_2)_{max} = N\left[\log(23.4) + (0.98336-1)\right]$$
$$= N(1.35258)$$
$$= 22.521 \text{ inches}$$

Using the same correction factor for the length, as given by Eq. (172), the new length becomes:

$$L_2 = N\left[\log(262) + (0.98336-1)\right]$$
$$= N(2.40166)$$
$$= 252.15 \text{ inches}$$

The ratio of $(y)_{max}$ of the major and minor curves is:

$$p = 26/20.8$$
$$= 1.25$$

and:

$$1/p = 0.8$$

Therefore, for the new body, from Eqs. (175) and (176):

$$(y)_{minor} = (2)/(1.25 + 1)(22.521)$$
$$= 20.018 \text{ inches}$$

$$(y)_{major} = (2)/(0.8 + 1)(22.521)$$
$$= 25.023 \text{ inches}$$

Correcting the equations of the new body for the change in ordinates, as was done in Section 5.60, we get:

$$y_{minor} = N\left[(0.23829) + (0.98336-1)\right](x)^{0.6} - N\left[(0.73174-4) + (0.98336-1)\right](x)^{2.05}$$
$$= N(0.22165)(x)^{0.6} - N(0.71510-4)(x)^{2.05}$$

$$y_{major} = N\left[(0.66636) + (0.98336-1)\right](x)^{0.4} - N\left[(0.28565-5) + (0.98336-1)\right](x)^{2.625}$$
$$= N(0.64972)(x)^{0.4} - N(0.26901-5)(x)^{2.625}$$

To correct the values of (x) for the change in length, the multiplier is again the reciprocal of the correction factor, raised to the exponent of (x), which is, in log form:

$$= (1.0000-1) - (0.98336-1)$$
$$= 0.01664$$

Substituting this in the above equations for $(y)_{minor}$ and $(y)_{major}$, there results:

$$y_{minor} = N\left[(0.22165) + (0.01664)^{0.6}\right](x)^{0.6} - N\left[(0.71510\text{-}4)\right.$$

$$\left. + (0.01664)^{2.05}\right](x)^{2.05}$$

$$= N(0.23163)(x)^{0.6} - N(0.74923\text{-}4)(x)^{2.05} \text{.....................} (177)$$

and:

$$y_{major} = N\left[(0.64972) + (0.01664)^{0.4}\right](x)^{0.4} - N\left[(0.26901\text{-}5)\right.$$

$$\left. + (0.01664)^{2.625}\right](x)^{2.625}$$

$$= N(0.65638)(x)^{0.4} - N(0.31269\text{-}5)(x)^{2.625} \text{.....................} (178)$$

Eqs. (177) and (178) are then the desired equations to give an elliptical body of volume equal to 231,000 cubic inches, with a length equal to 252.15 inches, and maximum ordinates, in two planes, of 20.018 and 25.023 inches.